MATH 1586 - EXAM 2 - SPRING 2018
SOLUTION

9 March 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (11 points) Use the ratio test to see if the series converges: $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{k!}$.

   Solution: Let $a_k = \dfrac{5^k}{k!}$ and calculate

$$\begin{aligned} \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{\frac{5^{k+1}}{(k+1)!}}{\frac{5^k}{k!}}\right| \\ &= \lim_{k\to\infty} \left|\frac{k!5^{k+1}}{5^k(k+1)!}\right| \\ &= \lim_{k\to\infty} \left|\frac{5}{k+1}\right| \\ &= 0, \end{aligned}$$

   and since $0 < 1$, we conclude from the ratio test that the series converges.

2. (14 points) If the series converges, then find its value. If it does not converge, explain why not.

   (a) (7 points) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{7^k}$

   Solution: The series converges because it is geometric with $r = \dfrac{1}{7}$. It converges to

$$\sum_{k=0}^{\infty} \frac{1}{7^k} = \frac{1}{1-\frac{1}{7}} = \frac{1}{\frac{6}{7}} = \frac{7}{6}.$$

   (b) (7 points) $\displaystyle\sum_{k=0}^{\infty} 1.2^k$

   Solution: The series diverges because it is geometric with $r = 1.2 > 1$.

3. (18 points) Calculate

   (a) (5 points) $\langle 1,0,2\rangle + 3\langle 2,1,0\rangle$
   Solution: Calculate

$$\langle 1,0,2\rangle + 3\langle 2,1,0\rangle = \langle 1,0,2\rangle + \langle 6,3,0\rangle = \langle 1+6, 0+3, 2+0\rangle = \langle 7,3,2\rangle.$$

   (b) (5 points) $\langle 1,0,2\rangle \cdot \langle 2,1,0\rangle$
   Solution: Calculate

$$\langle 1,0,2\rangle \cdot \langle 2,1,0\rangle = 1\cdot 2 + 0\cdot 1 + 2\cdot 0 = 2+0+0 = 2.$$

   (c) (8 points) $\langle 1,0,2\rangle \times \langle 2,1,0\rangle$
   Solution: Calculate

$$\begin{aligned} \langle 1,0,2\rangle \times \langle 2,1,0\rangle &= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \\ &= \vec{i}\det\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} - \vec{j}\det\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} + \vec{k}\det\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \vec{i}(0-2) - \vec{j}(0-4) + \vec{k}(1-0) \\ &= -2\vec{i} + 4\vec{j} + \vec{k} \\ &= \langle -2,4,1\rangle. \end{aligned}$$

4. (18 points) Find the position function ($\vec{r}(t)$) of a particle which has acceleration is $\vec{a}(t) = \langle 0, 1, 4\rangle$, an initial velocity of $\vec{v}(0) = \langle 1, 2, 3\rangle$, and at time $t = 1$ was at position $\vec{r}(1) = \langle 1, 1, 1\rangle$.
Solution: Integrate $\vec{a}$ to find $\vec{v}$:

$$\vec{v}(t) = \int \vec{a}(t)\mathrm{d}t = \int \langle 0, 1, 4\rangle \mathrm{d}t = \langle 0, t, 4t\rangle + \vec{c}_1.$$

Using the initial condition, we get

$$\underbrace{\langle 1, 2, 3\rangle}_{\text{given}} = \vec{v}(0) = \underbrace{\langle 0, 0, 0\rangle + \vec{c}_1}_{\text{calculated}},$$

therefore

$$\langle 1, 2, 3\rangle = \vec{c}_1.$$

Therefore we conclude that the velocity function is

$$\vec{v}(t) = \langle 1, t + 2, 4t + 3\rangle.$$

Integrate $\vec{v}$ to find $\vec{r}$ to get

$$\vec{r}(t) = \int \vec{v}(t)\mathrm{d}t = \int \langle 1, t + 2, 4t + 3\rangle \mathrm{d}t = \left\langle t, \frac{t^2}{2} + 2t, 2t^2 + 3t \right\rangle + \vec{c}_2.$$

Using the given value for $\vec{r}$, we get

$$\underbrace{\langle 1, 1, 1\rangle}_{\text{given}} = \vec{r}(1) = \underbrace{\left\langle 1, \frac{5}{2}, 5 \right\rangle}_{\text{calculated}} + \vec{c}_2.$$

Therefore

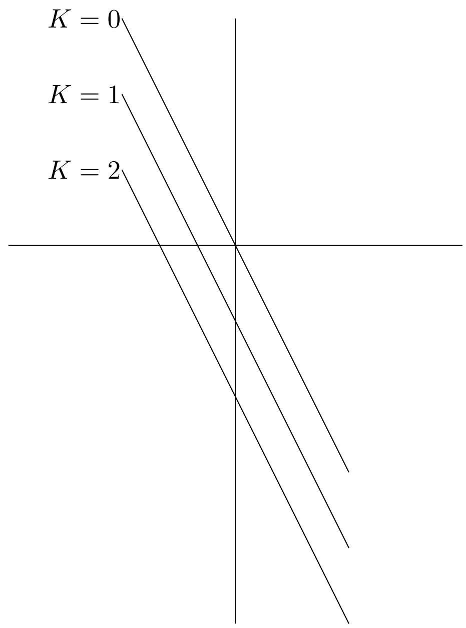
$$\vec{c}_2 = \langle 1, 1, 1\rangle - \left\langle 1, \frac{5}{2}, 5 \right\rangle = \left\langle 0, -\frac{3}{2}, -4 \right\rangle.$$

Therefore we conclude that the position function is

$$\vec{r}(t) = \left\langle t, \frac{t^2}{2} + 2t - \frac{3}{2}, 2t^2 + 3t - 4 \right\rangle$$

5. (16 points) Plot the level curves corresponding to $K = 0$, $K = 1$, and $K = 2$ of the given function.
   (a) (8 points) $f(x, y) = -2x - y$
   Solution: We first find equations for each level curve:

$\underline{K = 0}$	$\underline{K = 1}$	$\underline{K = 2}$
$0 = -2x - y$	$1 = -2x - y$	$2 = -2x - y$
$y = -2x$	$y = -2x - 1$	$y = -2x - 2$



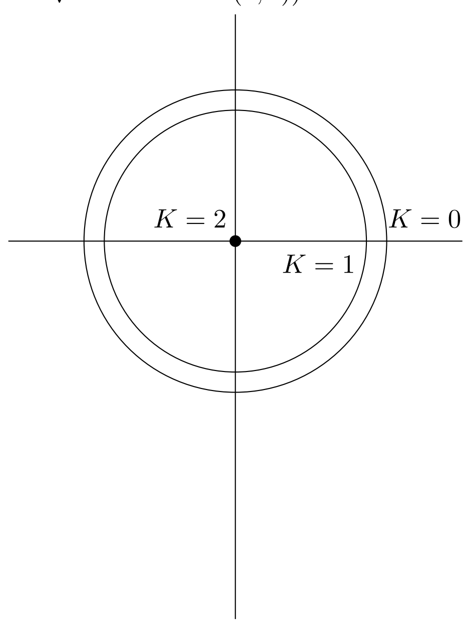
(b) (8 points) $f(x,y)=\sqrt{4-x^2-y^2}$.

Solution: We first find the equations for each level curve:

$\underline{K=0}$
$0=\sqrt{4-x^2-y^2}$
$x^2+y^2=4$ (the circle of radius 2 centered at $(0,0)$)

$\underline{K=1}$
$1=\sqrt{4-x^2-y^2}$
$x^2+y^2=3$ (the circle of radius $\sqrt{3}$ centered at $(0,0)$)

$\underline{K=2}$
$2=\sqrt{4-x^2-y^2}$
$x^2+y^2=0$ (a point at $(0,0)$)



6. (11 points) Show that the limit does not exist: $\lim_{(x,y)\to(0,0)} \frac{xy}{5x^2+2y^2}$.

Solution: First path (along x-axis – i.e. points of form $(x,y)=(x,0)$)
Along this path:

$$\lim_{(x,0)\to(0,0)} 0 = \lim_{x\to 0} 0 = 0.$$

Second path (along $y = x$ – i.e. points of the form $(x,y)=(x,x)$
Along this path:

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{7x^2} = \lim_{x\to 0} \frac{1}{7} = \frac{1}{7}.$$

Since we have found two paths whose limits have different values, we conclude that the limit does not exist.

7. (12 points) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   (a) (6 points) $f(x,y) = x^2y + 2x^3y^2$
   Solution: Calculate

$$\frac{\partial f}{\partial x} = 2xy + 6x^2y^2$$

   and

$$\frac{\partial f}{\partial y} = x^2 + 4x^3y$$

   (b) (6 points) $f(x,y) = ye^{xy}$
   Solution: Calculate

$$\frac{\partial f}{\partial x} = y^2e^{xy}$$

   and

$$\frac{\partial f}{\partial y} = e^{xy} + xye^{xy}$$