MATH 1586 - EXAM 1 - SPRING 2018
SOLUTION

9 February 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (18 points) Calculate

   (a) (9 points) $\int 5x^{17} + 3x^2 - x + 1 \mathrm{d}x$

   Solution: Calculate

$$\int 5x^{17} + 3x^2 - x + 1 \mathrm{d}x = \frac{5x^{18}}{18} + x^3 - \frac{x^2}{2} + x + C.$$

   (b) (9 points) $\int_3^8 w^2 + 3 \mathrm{d}w$

   Solution: Calculate

$$\begin{aligned} \int_3^8 w^2 + 3\mathrm{d}w &= \frac{w^3}{3} + 3w\Big|_3^8 \\ &= \left(\frac{8^3}{3} + 3(8)\right) - \left(\frac{3^3}{3} + 3(3)\right) \\ &= \left(\frac{512}{3} + 24\right) - \left(\frac{27}{3} + 9\right) \end{aligned}$$

2. (18 points) Calculate the integral using a u-substitution.

   (a) (9 points) $\int q^2 e^{q^3-1} \mathrm{d}q$

   Solution: Let $u = q^3 - 1$, then $\mathrm{d}u = 3q^2 \mathrm{d}u$, so $\frac{1}{3}\mathrm{d}u = q^2 \mathrm{d}q$. Thus, compute

$$\begin{aligned} \int q^2 e^{q^3-1} \mathrm{d}q &= \frac{1}{3}\int e^u \mathrm{d}u \\ &= \frac{1}{3}e^u + C \\ &= \frac{1}{3}e^{q^3-1} + C. \end{aligned}$$

   (b) (9 points) $\int \frac{1}{3z-4} \mathrm{d}z$

   Solution: Let $u = 3z - 4$ so $\mathrm{d}u = 3\mathrm{d}z$, hence $\frac{1}{3}\mathrm{d}u = \mathrm{d}z$. Compute

$$\begin{aligned} \int \frac{1}{3z-4} \mathrm{d}z &= \frac{1}{3}\int \frac{1}{u} \mathrm{d}u \\ &= \frac{1}{3}\ln(u) + C \\ &= \frac{1}{3}\ln(3z-4) + C. \end{aligned}$$

3. (10 points) Calculate using integration by parts

$$\int xe^{-2x} \mathrm{d}x.$$

Solution: Let $u = x$ and $\mathrm{d}v = e^{-2x}\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = -\frac{1}{2}e^{-2x}$. Therefore, by integration by parts,

$$\begin{aligned} \int xe^{-2x} \mathrm{d}x &= -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\mathrm{d}x \\ &= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

4. (10 points) Let $f(t) = e^{9t}$. Let $x > 9$. Use the definition of the Laplace transform to compute $\mathscr{L}\{f\}(x)$.
Solution: By definition,

$$\begin{aligned}
\mathscr{L}\{f\}(x) &= \int_0^\infty e^{9t}e^{-xt}\mathrm{d}t \\
&= \lim_{b\to\infty}\int_0^b e^{t(9-x)}\mathrm{d}t \\
&= \lim_{b\to\infty}\frac{1}{9-x}e^{t(9-x)}\Big|_0^b \\
&= \frac{1}{9-x}\lim_{b\to\infty}\left[e^{b(9-x)} - e^0\right] \\
&= \frac{1}{9-x}(0-1) \\
&= \frac{1}{x-9}.
\end{aligned}$$

5. (16 points) Recall Newton's law of cooling:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_0),$$

where T denotes the temperature at time t, k is a proportionality constant, and T_0 is the ambient temperature of the surrounding space.

When a cake is removed from an oven, its temperature is $250°F$. 5 minutes later, it is $230°F$. How long will it take for the cake to cool down to $90°F$ in a room that is $70°F$?
Solution: We are told that $T_0 = 70$ and two values: $T(0) = 250$ and $T(5) = 230$. With this information, we know that the differential equation is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - 70).$$

Separate variables in this equation to get

$$\frac{1}{T-70}\mathrm{d}T = k\mathrm{d}t.$$

Integrate both sides to get

$$\ln(T - 70) = kt + C.$$

Plug both sides into e^x to obtain

$$T - 70 = e^{k+C} = C_1e^{kt}, \quad C_1 = e^C.$$

Thus we have found

$$T = 70 + C_1e^{kt}.$$

To find C_1, use the condition $T(0) = 250$ to see

$$\underbrace{250}_{\text{given}} = T(0) = \underbrace{70 + C_1e^0}_{\text{calculated}}.$$

Thus since $e^0 = 1$, we get $180 = C_1$. Thus our model is

$$T = 70 + 180e^{kt}.$$

To find k, we use the condition $T(5) = 230$ to get

$$\underbrace{230}_{\text{given}} = T(5) = \underbrace{70 + 80e^{5k}}_{\text{calculated}}.$$

Solving this for k yields

$$\frac{1}{5}\ln\left(\frac{160}{180}\right) = \frac{1}{5}\ln\left(\frac{8}{9}\right) = k.$$

Thus our model is

$$T = 70 + 80e^{\frac{1}{5}\ln\left(\frac{8}{9}\right)t}.$$

Finally, we must find the time t for which $T(t) = 90$. This means we must solve

$$\underbrace{90}_{\text{given}} = T(t) = \underbrace{70 + 80e^{\frac{1}{5}\ln\left(\frac{8}{9}\right)t}}_{\text{calculated}}.$$

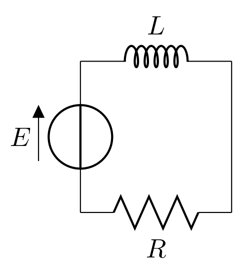
Solving for t yields

$$t = 5\frac{\ln\left(\frac{20}{80}\right)}{\ln\left(\frac{8}{9}\right)} \approx 58.84 \text{ minutes.}$$

6. (28 points) In a series circuit containing only a resistor (R, measured in ohms), an inductor (L, measured in henries which are ohm $\cdot$ sec), and a current at time t, $i(t)$ (measured in amps), Kirchoff's second law states that the sum of the voltage drop across the inductor $\left(L\frac{\mathrm{d}i}{\mathrm{d}t}\right)$ and the voltage drop across the resistor (iR) is the same as the impressed voltage ($E(t)$) on the circuit. We obtain the differential equation for the current $i(t)$

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri(t) = E(t),$$

where L and R are constants known as the inductance and resistance.



A 12-volt electromotive force is applied to a series circuit in which the inductance is 1 henry and the resistance is 8 ohms.

(a) (4 points) What is the differential equation we must solve here?

Solution: $\frac{\mathrm{d}i}{\mathrm{d}t} + 8i = 12$

(b) (7 points) Calculate $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{8t}i(t)\right]$.

Solution: Compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{8t}i(t)\right] = 8e^{8t}i(t) + e^{8t}\frac{\mathrm{d}i}{\mathrm{d}t}.$$

(c) (7 points) Multiply your differential equation on both sides by e^{8t} and then rewrite the left-hand side as $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{8t}i(t)\right]$.

Solution: We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{8t}i(t)\right] = 12e^{8t}.$$

(d) (10 points) Solve the differential equation by integrating and solving for $i(t)$.
(*note: we do not find a value of C here because no initial condition was given!*)

Solution: Integrating on the left cancels the $\frac{\mathrm{d}}{\mathrm{d}t}$ yielding

$$e^{8t}i(t) = 12\int e^{8t}\mathrm{d}t = \frac{12}{8}e^{8t} + C = \frac{3}{2}e^{8t} + C.$$

Thus we have

$$i(t) = \frac{3}{2} + Ce^{-8t}.$$

Formulas

If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Integration by parts:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Improper integral:

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\mathrm{d}x$$

Antiderivative of ln:

$$\int \ln(x)\mathrm{d}x = x\ln(x) - x + C$$

Laplace transform:

$$\mathscr{L}\{f\}(x) = \int_0^\infty f(t)e^{-xt}\mathrm{d}t$$