MATH 1540 - EXAM 2 - SPRING 2018
SOLUTION

8 March 2018
Instructor: Tom Cuchta

Instructions:

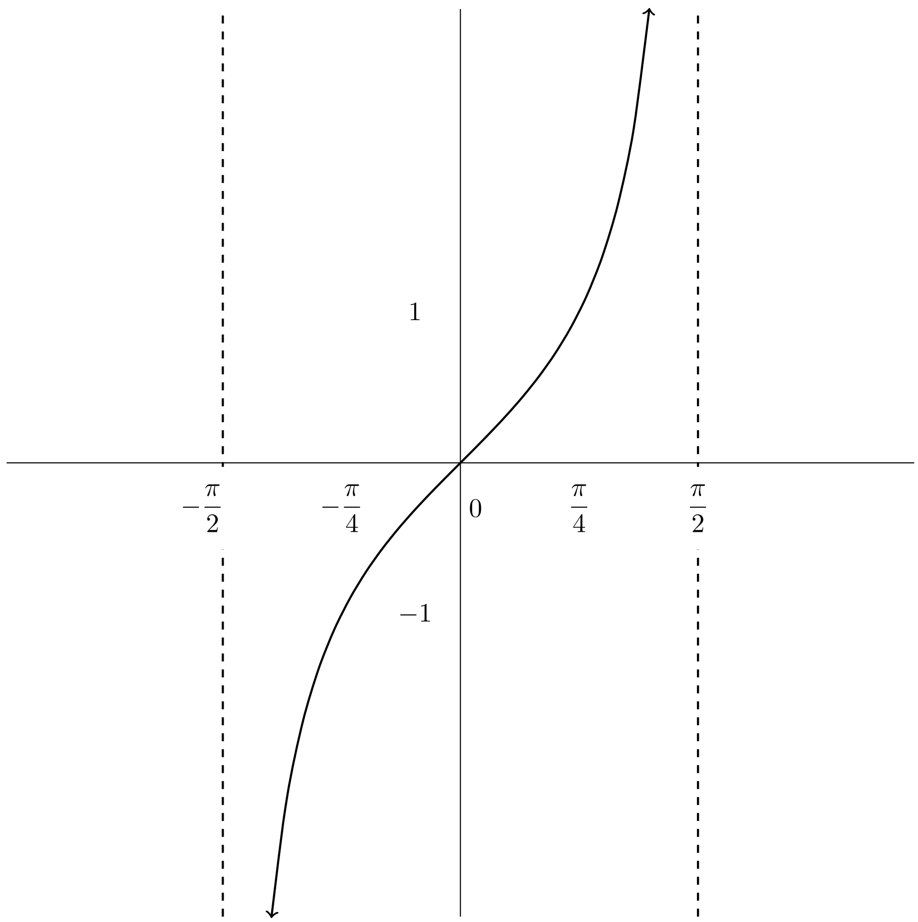
- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (25 points) Graph the function. Include all relevant labels on the x-axis and y-axis for full credit.

   (a) (5 points) Graph $y = \tan(x)$.

   Solution: There are asymptotes at $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. Therefore

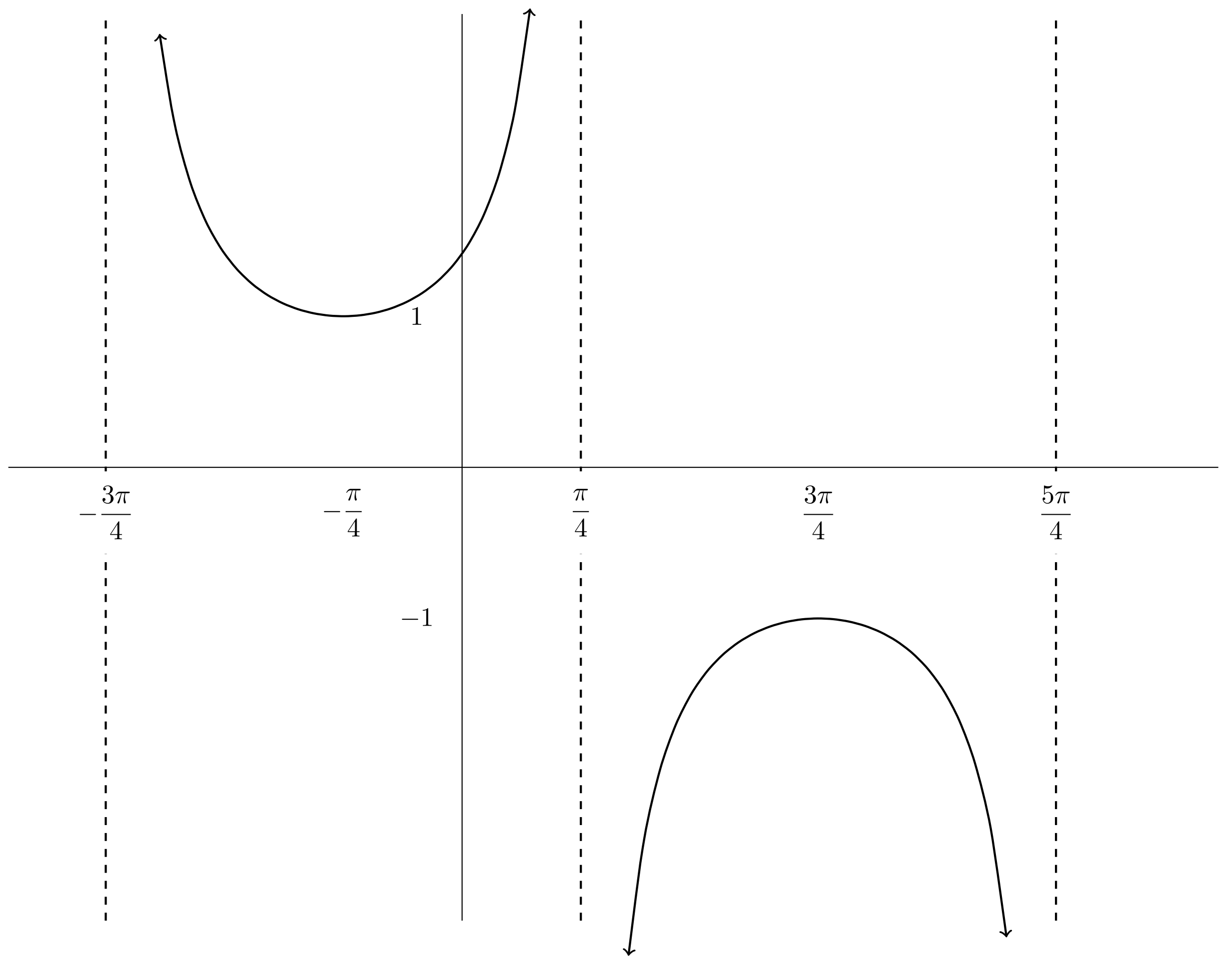
   Anchor points: $-\dfrac{\pi}{2}, -\dfrac{\pi}{4}, 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}$



(b) (10 points) Graph $y = \sec\left(x + \frac{\pi}{4}\right)$.

Solution: Secant has asymptotes at $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$, so:

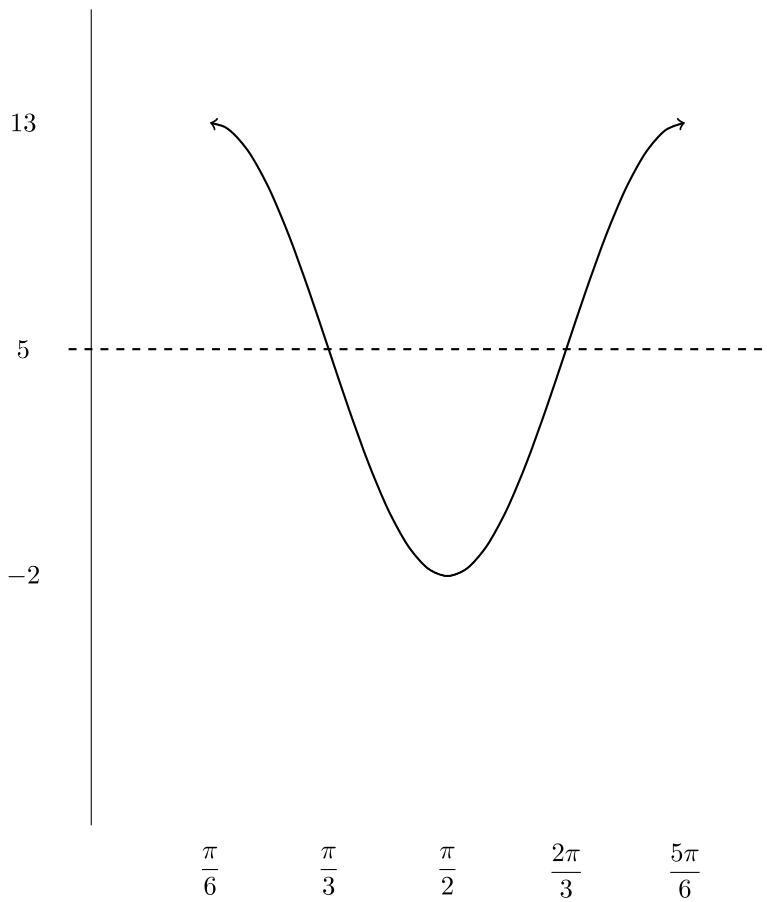
$$\textbf{Anchor points}: -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$
$$\downarrow \text{(h.shift)}$$
$$-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$$



(c) (10 points) Graph $y = 7\cos\left(3\left(x - \frac{\pi}{2}\right)\right) + 5$.

Solution: Start with the anchor points:

$$\textbf{Anchor points}: 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$
$$\downarrow \text{(h.shift)}$$
$$\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$$
$$\downarrow \text{(h.compr)}$$
$$\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$$



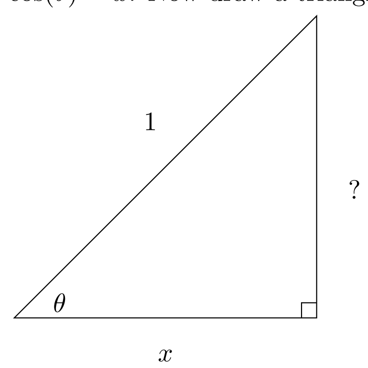
2. (24 points) Do each part.

(a) (8 points) Find the exact value of $\tan^{-1}(1)$.

Solution: $\tan^{-1}(1) = \dfrac{\pi}{4}$

(b) (8 points) Find the exact value of $\sin(\cos^{-1}(x))$.

Solution: Let $\theta = \cos^{-1}(x)$, then $\cos(\theta) = x$. Now draw a triangle for this situation:

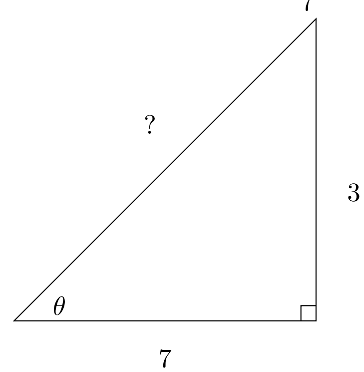


By the Pythagorean theorem, we know that $x^2 + ?^2 = 1^2$, hence $? = \sqrt{1-x^2}$. Therefore, we may compute

$$\sin\left(\cos^{-1}(x)\right) = \sin(\theta) = \sqrt{1-x^2}.$$

(c) (8 points) Find the exact value of $\csc\left(\tan^{-1}\left(-\dfrac{3}{7}\right)\right)$.

Solution: Let $\theta = \tan^{-1}\left(-\dfrac{3}{7}\right)$, so we may conclude that θ lives in QI or QIV. Then $\tan(\theta) = -\dfrac{3}{7}$, so we may conclude that θ lives in QII or QIV. Therefore θ is in QIV (so the cosecant of θ will be negative). Draw a triangle that agrees with $\tan(\theta) = -\dfrac{3}{7}$ (except for the sign):



To find ? use the Pythagorean theorem: $7^2 + 3^2 = ?^2$, hence $? = \sqrt{49+9} = \sqrt{58}$. Therefore compute

$$\csc\left(\tan^{-1}\left(-\frac{3}{7}\right)\right) = \csc(\theta) = -\frac{\sqrt{58}}{3}.$$

3. (24 points) (a) (8 points) Simplify $\dfrac{\cos(-x)\sec(-x)}{\sin(-x)\csc(-x)}$

Solution: Calculate

$$\begin{aligned}
\frac{\cos(-x)\sec(-x)}{\sin(-x)\csc(-x)} &= \frac{\cos(x)\frac{1}{\cos(-x)}}{-\sin(x)\frac{1}{\sin(-x)}} \\
&= \frac{\cos(x)\frac{1}{\cos(x)}}{-\sin(x)\frac{1}{-\sin(x)}} \\
&= \frac{1}{(-1)(-1)} \\
&= 1.
\end{aligned}$$

(b) (8 points) Write the first expression in terms of the second expression: $\dfrac{\cos(x)}{1+\sin(x)} + \tan(x); \cos(x)$

Solution: Calculate

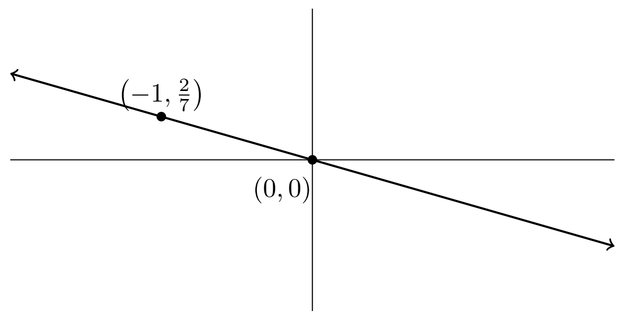
$$\begin{aligned}
\frac{\cos(x)}{1+\sin(x)} + \tan(x) &= \frac{\cos(x)}{1+\sin(x)} + \frac{\sin(x)}{\cos(x)} \\
&= \frac{\cos^2(x) + \sin(x)(1+\sin(x))}{\cos(x)(1+\sin(x))} \\
&= \frac{\sin(x) + \overbrace{\cos^2(x) + \sin^2(x)}^{=1}}{\cos(x)(1+\sin(x))} \\
&= \frac{\sin(x)+1}{\cos(x)(1+\sin(x))} \\
&= \frac{1}{\cos(x)}.
\end{aligned}$$

(c) (8 points) Prove the following identity: $\cos(x)(\tan(x) - \sec(-x)) = \sin(x) - 1$.
Solution: Start with the left:

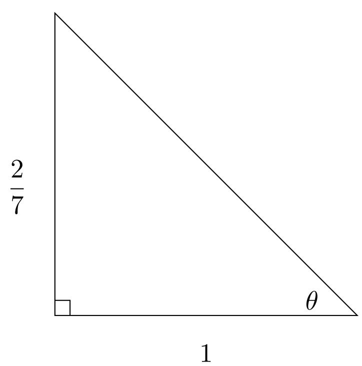
$$\begin{aligned}
\cos(x)(\tan(x) - \sec(-x)) &= \cos(x)\left[\frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(-x)}\right] \\
&= \cos(x)\left[\frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)}\right] \\
&= \sin(x) - 1.
\end{aligned}$$

4. (3 points) Calculate, if possible. If not possible, explain why not ("my calculator says so" is not an explanation). Express your answer correct to two decimal points.
   (a) (1 point) $\sin^{-1}(0.783)$
   Solution: $\sin^{-1}(0.783) \approx 0.899$ radians $= 51.54°$
   (b) (1 point) $\cos^{-1}(2)$
   Solution: This is not possible, because 2 is not in the domain of $\cos^{-1}$. Asking the question "what is $\cos^{-1}(2)$?" is the same as asking the question "what angle in the unit circle gives us a cosine equal to 2?" — there isn't one!
   (c) (1 point) $\tan^{-1}\left(\frac{\pi}{2}\right)$
   Solution: $\arctan\left(\frac{\pi}{2}\right) \approx 1.003$ radians $= 57.52°$

5. (12 points) The line $y = -\frac{2}{7}x$ passes through the origin in the xy-plane. What is the angle that the line makes with the negative x-axis? Express your answer correct to two decimal places.
   Solution:



There is a triangle here – draw it:



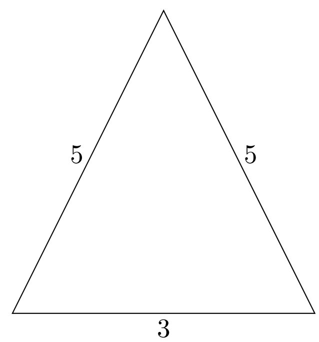
From this picture, we observe that

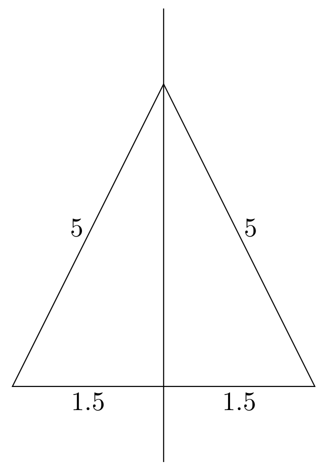
$$\tan(\theta) = \frac{\frac{2}{7}}{1} = \frac{2}{7}.$$

Therefore,

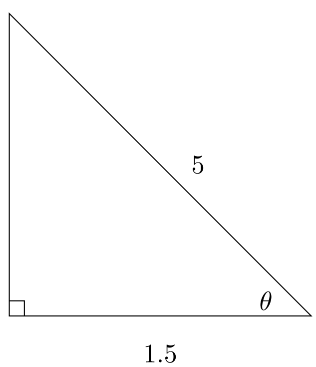
$$\theta = \tan^{-1}\left(\frac{2}{7}\right) \approx 0.27 \text{ radians} = 15.95°.$$

6. (12 points) An isosceles triangle has two sides of length 5 inches. The third side has a length 3 inches. Find the angle that a side of length 5 makes with the side of length 3. Express your answer correct to two decimal places.
   Solution: Draw the described triangle:





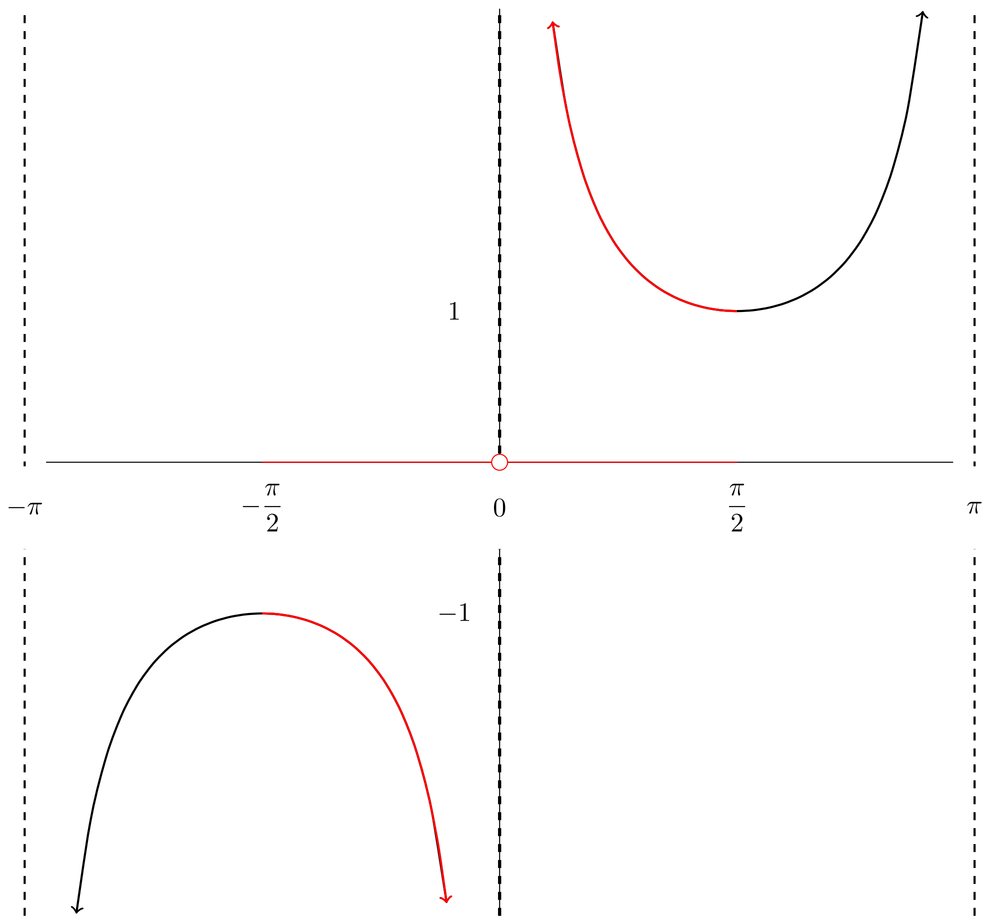
Take, say, the right triangle.



From this we see that $\cos(\theta) = \dfrac{1.5}{5}$, therefore

$$\theta = \cos^{-1}\left(\frac{1.5}{5}\right) \approx 1.266 \text{ radians} = 72.54^\circ.$$

7. (4 points) (**Bonus**): Draw $y = \csc(x)$. Indicate an interval in the x-axis which could serve as a domain restriction of $\csc(x)$ that could be used to define an inverse cosecant function.
(*hint: recall "one-to-one" and what it has to do with inverse functions*)
Solution:



In order to be one-to-one, a domain restriction of cosecant like $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is necessary. This defines a $\csc^{-1}$ function whose domain is $(-\infty, -1] \cup [1, \infty) = \mathbb{R} \setminus \{(-1, 1)\}$ and whose range is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.