Homework 5

Consider the following table that relates characters to numbers to integers:

A	1	B	2	C	3	D	4	E	5	F	6
G	7	H	8	I	9	J	10	K	11	L	12
M	13	N	14	O	15	P	16	Q	17	R	18
S	19	T	20	U	21	V	22	W	23	X	24
Y	25	Z	26	`<space>`	27						

We will take a message to be encrypted and arrange the message into an $m \times m$ matrix of an appropriate size, adding spaces at the end to make it square. Let A be an $m \times m$ invertible matrix. We defined the matrix E ("encrypted") by the formula $E = MA$.

Consider the following message: "`ZUGZWANG FATIGUE`". This message contains 16 characters (including the space as a character). We can arrange this message into a 4×4 matrix (note: if I had, say, 17 characters, then I would put 8 spaces at the end to make it 25 characters long and then use a 5×5 matrix).

Therefore I can encode the message in the following matrix (using my assignment of letters):

$$M = \begin{bmatrix} \mathtt{Z} & \mathtt{U} & \mathtt{G} & \mathtt{Z} \\ \mathtt{W} & \mathtt{A} & \mathtt{N} & \mathtt{G} \\ \_\_ & \mathtt{F} & \mathtt{A} & \mathtt{T} \\ \mathtt{I} & \mathtt{G} & \mathtt{U} & \mathtt{E} \end{bmatrix} \overset{\text{table}}{=} \begin{bmatrix} 26 & 21 & 7 & 26 \\ 23 & 1 & 14 & 7 \\ 27 & 6 & 1 & 20 \\ 9 & 7 & 21 & 5 \end{bmatrix}$$

Define the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. This matrix is invertible with $A^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$ (use wolframalpha.com to convince yourself of this if you don't want to do it by hand).

Then the encrypted matrix for our message is

$$E = MA = \begin{bmatrix} 26 & 21 & 7 & 26 \\ 23 & 1 & 14 & 7 \\ 27 & 6 & 1 & 20 \\ 9 & 7 & 21 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 80 & 26 & 73 & 28 \\ 45 & 7 & 31 & 15 \\ 54 & 20 & 53 & 7 \\ 42 & 5 & 21 & 28 \end{bmatrix}.$$

This matrix has some numbers we did not assign previously, however it does yield the following encrypted message according to our assigned alphabet:

$$E_{\text{encoded}} = (80)Z(73)(28)(45)G(31)O(54)T(53)G(42)EO(28),$$

where (#) stands in for an unassigned value. Clearly we need to have a more robust alphabet. Now suppose you are given the message E_{encoded}. How do you deencrypt it? You arrange the message into a 4×4 matrix and then multiply, on the RIGHT, by the matrix A^{-1}: compute

$$EA^{-1} = (MA)A^{-1} = \begin{bmatrix} 80 & 26 & 73 & 28 \\ 45 & 7 & 31 & 15 \\ 54 & 20 & 53 & 7 \\ 42 & 5 & 21 & 28 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 26 & 21 & 7 & 26 \\ 23 & 1 & 14 & 7 \\ 27 & 6 & 1 & 20 \\ 9 & 7 & 21 & 5 \end{bmatrix} = M,$$

which is the original matrix M we started with.

1. (5 points) Use the following alphabet assignment and the previous matrix A^{-1} to decode the following encrypted message (yes using software is ok for this):

$$\alpha\ \Omega J \delta W \mu \Xi \xi E N R \psi Y \Phi E$$

A	1	B	2	C	3	D	4	E	5	F	6
G	7	H	8	I	9	J	10	K	11	L	12
M	13	N	14	O	15	P	16	Q	17	R	18
S	19	T	20	U	21	V	22	W	23	X	24
Y	25	Z	26	<space>	27	α	56	δ	98	Ξ	54
ψ	49	μ	71	ξ	28	Φ	48	Ω	51		

Solution: Using the table, we convert the message into an ecrypted matrix:

$$E = \begin{bmatrix} \alpha & & \Omega & J \\ \delta & W & \mu & \Xi \\ \xi & E & N & R \\ \psi & Y & \Phi & E \end{bmatrix} \overset{\text{table}}{=} \begin{bmatrix} 56 & 27 & 51 & 10 \\ 98 & 23 & 71 & 54 \\ 28 & 5 & 14 & 18 \\ 49 & 25 & 48 & 5 \end{bmatrix}.$$

To recover the message, multiply on the right by the matrix A^{-1} and translate through the table: compute

$$EA^{-1} = \begin{bmatrix} 56 & 27 & 51 & 10 \\ 98 & 23 & 71 & 54 \\ 28 & 5 & 14 & 18 \\ 49 & 25 & 48 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 19 & 5 & 5 & 27 \\ 21 & 27 & 27 & 23 \\ 5 & 4 & 14 & 5 \\ 19 & 4 & 1 & 25 \end{bmatrix} \overset{\text{table}}{=} \begin{bmatrix} S & E & E & \\ U & & & W \\ E & D & N & E \\ S & D & A & Y \end{bmatrix}.$$

And we see the recovered message is `SEE U WEDNESDAY`.

2. (5 points) See the notes at the following address for an example of a problem like this one and relevant definitions from chemistry: http://tomcuchta.com/teach/classes/2015/MATH-3108-1A1B-Spring-2015-MissouriST/notes/chemistry.php

Use a vector equation (not another way) to balance the following unbalanced chemical equation in which xenon diflouride combines with water to produce oxygen gas, xenon gas, and hydrofluoric acid:

$$\mathrm{XeF_2} + \mathrm{H_2O} \to \mathrm{O_2} + \mathrm{Xe} + \mathrm{HF}.$$

Solution: We represent each molecule as a vector of length 4 with the first row being number of xenon atoms, the second row the number of fluorine atoms, the third row the number of hydrogen atoms, and the fourth row the number of oxygen atoms. Hence Xe F H O

$$\mathrm{XeF_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \mathrm{H_2O} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \mathrm{O_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \mathrm{Xe} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathrm{HF} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

To balance the equation means to find a solution to the vector equation

$$(*) \qquad c_1\mathrm{XeF_2} + c_2\mathrm{H_2O} + c_3\mathrm{O_2} + c_4\mathrm{Xe} + c_5\mathrm{HF} = \vec{0}.$$

Write the augmented matrix and compute

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix},$$

which encodes the following system of equations:

$$\begin{cases} c_1 + \frac{1}{2}c_5 = 0 \\ c_2 + \frac{1}{2}c_5 = 0 \\ c_3 - \frac{1}{4}c_5 = 0 \\ c_4 - \frac{1}{2}c_5 = 0. \end{cases}$$

Therefore we see that we have the following solution with free variable c_5:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = c_5 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

It is customary to choose a value for c_5 that makes the fractions all integers, so choose $c_5 = -4$ to get the following "particular solution" (why negative 4 and not positive 4? I want the coefficients of XeF_2 and H_2O to be positive)

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ -2 \\ -4 \end{bmatrix}.$$

In other words, $(*)$ becomes

$$2\mathrm{XeF}_2 + 2\mathrm{H}_2\mathrm{O} + (-1)\mathrm{O}_2 + (-2)\mathrm{Xe} + (-4)\mathrm{HF} = \vec{0},$$

which yields the balanced chemical equation

$$2\mathrm{XeF}_2 + 2\mathrm{H}_2\mathrm{O} \longrightarrow \mathrm{O}_2 + 2\mathrm{Xe} + 4\mathrm{HF}.$$