Homework 5

Consider the following table that relates characters to numbers to integers:

A	1	B	2	C	3	D	4	E	5	F	6
G	7	H	8	I	9	J	10	K	11	L	12
M	13	N	14	O	15	P	16	Q	17	R	18
S	19	T	20	U	21	V	22	W	23	X	24
Y	25	Z	26	<space>	27						

We will take a message to be encrypted and arrange the message into an $m \times m$ matrix of an appropriate size, adding spaces at the end to make it square. Let A be an $m \times m$ invertible matrix. We defined the matrix E ("encrypted") by the formula $E = MA$.

Consider the following message: "`ZUGZWANG FATIGUE`". This message contains 16 characters (including the space as a character). We can arrange this message into a 4×4 matrix (note: if I had, say, 17 characters, then I would put 8 spaces at the end to make it 25 characters long and then use a 5×5 matrix).

Therefore I can encode the message in the following matrix (using my assignment of letters):

$$M = \begin{bmatrix} \texttt{Z} & \texttt{U} & \texttt{G} & \texttt{Z} \\ \texttt{W} & \texttt{A} & \texttt{N} & \texttt{G} \\ \underline{\quad} & \texttt{F} & \texttt{A} & \texttt{T} \\ \texttt{I} & \texttt{G} & \texttt{U} & \texttt{E} \end{bmatrix} \overset{\text{table}}{=} \begin{bmatrix} 26 & 21 & 7 & 26 \\ 23 & 1 & 14 & 7 \\ 27 & 6 & 1 & 20 \\ 9 & 7 & 21 & 5 \end{bmatrix}$$

Define the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. This matrix is invertible with $A^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$ (use wolframalpha.com to convince yourself of this if you don't want to do it by hand).

Then the encrypted matrix for our message is

$$E = MA = \begin{bmatrix} 26 & 21 & 7 & 26 \\ 23 & 1 & 14 & 7 \\ 27 & 6 & 1 & 20 \\ 9 & 7 & 21 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 80 & 26 & 73 & 28 \\ 45 & 7 & 31 & 15 \\ 54 & 20 & 53 & 7 \\ 42 & 5 & 21 & 28 \end{bmatrix}.$$

This matrix has some numbers we did not assign previously, however it does yield the following encrypted message according to our assigned alphabet:

$$E_{\text{encoded}} = (80)Z(73)(28)(45)G(31)O(54)T(53)G(42)EO(28),$$

where (#) stands in for an unassigned value. Clearly we need to have a more robust alphabet. Now suppose you are given the message E_{encoded}. How do you deencrypt it? You arrange the message into a 4×4 matrix and then multiply, on the RIGHT, by the matrix A^{-1}: compute

$$EA^{-1} = (MA)A^{-1} = \begin{bmatrix} 80 & 26 & 73 & 28 \\ 45 & 7 & 31 & 15 \\ 54 & 20 & 53 & 7 \\ 42 & 5 & 21 & 28 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 26 & 21 & 7 & 26 \\ 23 & 1 & 14 & 7 \\ 27 & 6 & 1 & 20 \\ 9 & 7 & 21 & 5 \end{bmatrix} = M,$$

which is the original matrix M we started with.

1. (5 points) Use the following alphabet assignment and the previous matrix A^{-1} to decode the following encrypted message (yes using software is ok for this):

$$\alpha \ \Omega J \delta W \mu \Xi \xi E N R \psi Y \Phi E$$

A	1	B	2	C	3	D	4	E	5	F	6
G	7	H	8	I	9	J	10	K	11	L	12
M	13	N	14	O	15	P	16	Q	17	R	18
S	19	T	20	U	21	V	22	W	23	X	24
Y	25	Z	26	`<space>`	27	α	56	δ	98	Ξ	54
ψ	49	μ	71	ξ	28	Φ	48	Ω	51		

2. (5 points) See the notes at the following address for an example of a problem like this one and relevant definitions from chemistry: http://tomcuchta.com/teach/classes/2015/MATH-3108-1A1B-Spring-2015-MissouriST/notes/chemistry.php

   Use a vector equation (not another way) to balance the following unbalanced chemical equation in which xenon diflouride combines with water to produce oxygen gas, xenon gas, and hydrofluoric acid:

$$\mathrm{XeF_2 + H_2O \rightarrow O_2 + Xe + HF}.$$