MATH 3362 - EXAM 3 - FALL 2017
SOLUTION

Friday 17 November 2017
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (8 points) Consider the vector space $(\mathscr{P}_2, \mathbb{R})$. Find the coordinate vector of $p(x) = x^2 - 3$ with respect to the basis $\mathcal{B} = \{1, 1 - x, 1 + x^2\}$ (i.e. find $[p]_{\mathcal{B}}$).

Solution: The coordinate vector is given by $[x^2 - 3]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, where c_1, c_2, c_3 solve the vector equation

$$c_1(1) + c_2(1 - x) + c_3(1 + x^2) = x^2 - 3.$$

Simplify the left-hand side to get

$$(c_1 + c_2 + c_3)(1) + (-c_2)x + (c_3)x^2 = x^2 - 3.$$

Therefore $c_3 = 1$, $c_2 = 0$, and $c_1 + c_2 + c_3 = -3$ hence $c_1 + 1 = -3$, so $c_1 = -4$. Thus the coordinate vector is $[x^2 - 3]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$.

2. (24 points) Define $T\colon \mathbb{R}^{2\times 1} \to \mathbb{R}^{2\times 1}$ by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$.

(a) (12 points) Is T one-to-one?

Solution: Assume that $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$. This means

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix},$$

so

$$\begin{bmatrix} a + 2b \\ 3a + 4b \end{bmatrix} = \begin{bmatrix} c + 2d \\ 3c + 4d \end{bmatrix}.$$

Solve this by writing an augmented matrix and putting it into reduced echelon form:

$$\begin{bmatrix} 1 & 2 & c + 2d \\ 3 & 4 & 3c + 4d \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & c + 2d \\ 0 & -2 & -2d \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}.$$

Hence $a = c$ and $b = d$. Therefore T is one-to-one.

(b) (12 points) Is T onto?

Solution: Let $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{R}^{2\times 1}$ and consider the equation $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. This is

$$\begin{bmatrix} a+2b \\ 3a+4b \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Solve this equation:

$$\begin{bmatrix} 1 & 2 & \alpha \\ 3 & 4 & \beta \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & \alpha \\ 0 & -2 & \beta - 3\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \beta - 2\alpha \\ 0 & 1 & \frac{3}{2}\alpha - \frac{\beta}{2} \end{bmatrix}.$$

3. (8 points) Let $V = \{p \in \mathscr{P}_1 \colon p(0) = 0\}$ and let $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} \colon b = c = d \right\}$. Is the vector space $(V, \mathbb{R})$ isomorphic to the vector space $(W, \mathbb{R})$? Why or why not?
Solution: A polynomial $p = ax + b \in V$ provided that $p(0) = b = 0$. Therefore $V = \{ax \colon a \in \mathbb{R}\} = \operatorname{span}\{1\}$. Hence $\dim((V, \mathbb{R})) = 1$. A matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ provided that $b = c = d$ so we see

$$W = \left\{ \begin{bmatrix} a & b \\ b & b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

Therefore $\dim((W, \mathbb{R})) = 2$. Since these vector spaces don't have the same dimension, they are **not** isomorphic.

4. (23 points) (Chebyshev polynomials) Define a linear transformation $T \colon \mathscr{P}_2 \to \mathscr{P}_2$ by

$$T(p) = (1 - x^2)p'' - xp' + 4p = 0.$$

(a) (11 points) Find $\ker(T)$.
Solution: Let $p = ax^2 + bx + c$ then $p' = 2ax + b$ and $p'' = 2a$. Substituting these into the equation $T(p) = 0$ yields

$$(1 - x^2)(2a) - x(2ax + b) + 4(ax^2 + bx + c) = 0.$$

Combining like-terms on the left yields

$$(-2a - 2a + 4a)x^2 + (-b + 4b)x + (2a + 4c) = 0.$$

Hence $b = 0$ and $a = -2c$. Therefore

$$\ker(T) = \{(-2c)x^2 + c \colon c \in \mathbb{R}\} = \operatorname{span}\{-2x^2 + 1\}.$$

(b) (6 points) Use your answer from a.) to determine $\operatorname{nullity}(T)$.
Solution: In part a.) we see that a basis for $\ker(T)$ is $\operatorname{span}\{-2x^2 + 1\}$, and so

$$\operatorname{nullity}(T) = \dim(\ker(T)) = 1.$$

(c) (6 points) Use your answer from b.) and the Rank-Nullity theorem to determine rank(T).
Solution: We know that $\dim(\mathscr{P}_2) = 3$. The Rank-Nullity theorem says that

$$\text{rank}(T) + \text{nullity}(T) = \dim(\mathscr{P}_2).$$

Plugging in what we know: $\text{rank}(T) + 1 = 3$, hence $\text{rank}(T) = 2$.

5. (17 points) Consider the inner product space $(\mathscr{P}_2, \mathbb{R}, \langle\cdot,\cdot\rangle)$, where $\langle p, q\rangle = \int_0^1 p(x)q(x)x\mathrm{d}x$.

(a) (11 points) Calculate $\langle 1, x^2 - x^4\rangle$. (Recall that integration is a linear transformation and that $\int_0^1 x^n \mathrm{d}x = \frac{1}{n+1}$)
Solution: Compute

$$\langle 1, x^2 - x^4\rangle = \int_0^1 (1)(x^2 - x^4)x\mathrm{d}x = \int_0^1 x^3\mathrm{d}x - \int_0^1 x^5\mathrm{d}x = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

(b) (6 points) From your calculation in part a.), is the polynomial 1 orthogonal to the polynomial $x^2 - x^4$?
Solution: No – to be orthogonal it would mean that the inner product equalled zero.

6. (20 points) (Jacobi polynomials $P_n^{(1,1)}$) Consider the inner product space $(\mathscr{P}, \mathbb{R}, \langle p, q\rangle)$ where

$$\langle p, q\rangle = \int_{-1}^1 (1-x)(1+x)p(x)q(x)\mathrm{d}x.$$

It can be shown that $\langle 1, 1\rangle = \frac{4}{3}$, $\langle 1, x\rangle = 0$, $\langle 1, x^2\rangle = \frac{4}{15}$, and $\langle 1, x^3\rangle = 0$. Use the Gram-Schmidt process to orthogonalize the following set of vectors: $\{1, x, x^2\}$.
Solution: Orthogonalize the set by letting $\vec{u}_1 = 1$ and then

$$\vec{u}_2 = \vec{v}_1 - \text{proj}_{\vec{u}_1}(\vec{v}_1) = x - \frac{\langle 1, x\rangle}{\langle 1, 1\rangle}(1) = x,$$

and

$$\begin{aligned}
\vec{u}_3 &= \vec{v}_3 - \text{proj}_{\vec{u_1}}(\vec{v}_3) - \text{proj}_{\vec{u_2}}(\vec{v}_3) \\
&= x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x, x^2\rangle}{\langle x, x\rangle}x \\
&= x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle 1, x^3\rangle}{\langle 1, x^2\rangle}x \\
&= x^2 - \frac{\frac{4}{15}}{\frac{4}{3}} - 0x \\
&= x^2 - \frac{3}{15} \\
&= x^2 - \frac{1}{5}.
\end{aligned}$$