MATH 3362 - EXAM 1 FALL 2017
SOLUTION

Friday 15 September 2017
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (8 points) Circle **L** for linear and **N** for nonlinear.

   (a) (2 points) (**L**) **N** $0.1x_1 - 0.5x_2 = 0$

   (b) (2 points) **L** (**N**) $5\cos(x) + y + 3z = 5$

   (c) (2 points) (**L**) **N** $2x + 3y + z = 0$

   (d) (2 points) (**L**) **N** $\cos(5)x + y + 3z = 5$

2. (8 points) Compute the linear combination.

   (a) (4 points) (Field: $\mathbb{R}$) $2\begin{bmatrix}1\\0\\2\end{bmatrix} + 3\begin{bmatrix}\pi\\-5\\\sqrt{2}\end{bmatrix}$

   Solution: Compute

$$\begin{aligned} 2\begin{bmatrix}1\\0\\2\end{bmatrix} + 3\begin{bmatrix}\pi\\-5\\\sqrt{2}\end{bmatrix} &= \begin{bmatrix}2\\0\\4\end{bmatrix} + \begin{bmatrix}3\pi\\-15\\3\sqrt{2}\end{bmatrix} \\ &= \begin{bmatrix}2+3\pi\\-15\\4+3\sqrt{2}\end{bmatrix}. \end{aligned}$$

   (b) (4 points) (Field: $\mathbb{C}$) $i\begin{bmatrix}2i & 3\\8 & -i\end{bmatrix} + 3\begin{bmatrix}0 & 2\\4 & 0\end{bmatrix}$

   Solution: Compute

$$\begin{aligned} i\begin{bmatrix}2i & 3\\8 & -i\end{bmatrix} + 3\begin{bmatrix}0 & 2\\4 & 0\end{bmatrix} &= \begin{bmatrix}-2 & 3i\\8i & 1\end{bmatrix} + \begin{bmatrix}0 & 6\\12 & 0\end{bmatrix} \\ &= \begin{bmatrix}-2 & 6+3i\\12+8i & 1\end{bmatrix}. \end{aligned}$$

3. (21 points) Put the following matrices into reduced echelon form.

   (a) (7 points) (Field: $\mathbb{R}$) $\begin{bmatrix}1 & 0 & 2\\0 & 10 & 1\\1 & 5 & -1\end{bmatrix}$

   Solution: Compute

$$\begin{aligned} \begin{bmatrix}1 & 0 & 2\\0 & 10 & 1\\1 & 5 & -1\end{bmatrix} &\overset{r_3^*=r_3-r_1}{\sim} \begin{bmatrix}1 & 0 & 2\\0 & 10 & 1\\0 & 5 & -3\end{bmatrix} \\ &\overset{r_3^*=r_3-\frac{1}{2}r_2}{\sim} \begin{bmatrix}1 & 0 & 2\\0 & 10 & 1\\0 & 0 & -\frac{7}{2}\end{bmatrix} \\ &\overset{r_2^*=\frac{1}{10}r_2, r_3^*=-\frac{2}{7}r_3}{\sim} \begin{bmatrix}1 & 0 & 2\\0 & 1 & \frac{1}{10}\\0 & 0 & 1\end{bmatrix} \\ &\overset{r_2^*=r_2-\frac{1}{10}r_3, r_1^*=r_1-2r_3}{\sim} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}. \end{aligned}$$

   (b) (7 points) (Field: $\mathbb{Z}_3$) $\begin{bmatrix}0 & 2 & 1\\1 & 1 & 2\\0 & 1 & 2\end{bmatrix}$

Solution: Compute

$$\begin{aligned}
\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} &\overset{r_1 \leftrightarrow r_2}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\
&\overset{r_3^* = r_3 - 2r_2}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
&\overset{r_1 = r_1 + r_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
&\overset{r_2^* = 2r_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

(c) (7 points) (Field: $\mathbb{C}$) $\begin{bmatrix} i & 1+i & 0 \\ 1 & 2i & 1 \end{bmatrix}$

Solution: Compute

$$\begin{aligned}
\begin{bmatrix} i & 1+i & 0 \\ 1 & 2i & 1 \end{bmatrix} &\overset{r_2^* = r_2 + ir_1}{\sim} \begin{bmatrix} i & 1+i & 0 \\ 0 & 3i-1 & 1 \end{bmatrix} \\
&\overset{r_2^* = \frac{1}{3i-1} r_2}{\sim} \begin{bmatrix} i & 1+i & 0 \\ 0 & 1 & \frac{1}{3i-1} \end{bmatrix} \\
&\overset{r_1^* = r_1 - (1+i) r_2}{\sim} \begin{bmatrix} i & 0 & -\frac{1+i}{3i-1} \\ 0 & 1 & \frac{1}{3i-1} \end{bmatrix} \\
&\overset{r_1^* = \frac{1}{i} r_1}{\sim} \begin{bmatrix} 1 & 0 & -\frac{1}{i}\left(\frac{1+i}{3i-1}\right) \\ 0 & 1 & \frac{1}{3i-1} \end{bmatrix} \\
&\overset{r_1^* = \frac{1}{i} r_1}{\sim} \begin{bmatrix} 1 & 0 & \frac{1+i}{i+3} \\ 0 & 1 & \frac{1}{3i-1} \end{bmatrix}
\end{aligned}$$

Note: the above is an acceptable answer. However, the complex numbers could also be put into "standard form $a + bi$" by computing

$$\frac{1+i}{3+i} = \frac{1+i}{3+i}\left(\frac{3-i}{3-i}\right) = \frac{3 - i + 3i - i^2}{9+1} = \frac{4+2i}{10} = \frac{2}{5} + \frac{1}{5}i$$

and

$$\frac{1}{-1+3i} = \frac{1}{-1+3i}\left(\frac{-1-3i}{-1-3i}\right) = \frac{-1-3i}{1+9} = \frac{-1-3i}{10} = -\frac{1}{10} - \frac{3}{10}i,$$

yielding

$$\begin{bmatrix} i & 1+i & 0 \\ 1 & 2i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{5} + \frac{1}{5}i \\ 0 & 1 & -\frac{1}{10} - \frac{3}{10}i \end{bmatrix}.$$

4. (21 points) Answer the following questions.

(a) (7 points) (Field: $\mathbb{R}$) Write the vector $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors $\vec{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ **or** explain why that is not possible.

Solution: We can determine the answer by solving the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b},$$

i.e.

$$c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Write the augmented matrix and row reduce: compute

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \overset{r_3^*=r_3+r_1}{\sim} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$
$$\overset{r_3^*=r_3-2r_2}{\sim} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$\overset{r_1^*=-r_1, r_3^*=\frac{1}{2}r_2}{\sim} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\overset{r_1^*=r_1+2r_2}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\overset{r_1^*=r_1+r_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From this we conclude that there is no solution because it encodes the following system of equations:

$$\begin{cases} c_1 & & = 0 \\ & c_2 & = 0 \\ & 0 & = 1. \end{cases}$$

The third equation is always false, and hence no solution exists.

(b) (7 points) (Field: $\mathbb{R}$) Is the vector $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the span of the columns of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$?

Solution: Recall that the span of a set of vectors is the set of all linear combinations of those vectors. Therefore, this question is asking whether or not the following vector equation has a solution:

$$c_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

To answer the question, set up the augmented matrix and compute

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix} \overset{r_2^*=r_2-4r_1, r_3^*=r_3-7r_1}{\sim} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{bmatrix}$$
$$\overset{r_2^*-\frac{1}{3}r_2, r_3^*=-\frac{1}{6}r_3}{\sim} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$
$$\overset{r_3^*=r_3-r_2}{\sim} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\overset{r_1^*=r_1-2r_2}{\sim} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) (7 points) (Field: $\mathbb{R}$) Prove that $\mathbb{R}^{2\times 1} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$.

Solution: Recall that a set X equals a set Y if and only if $X \subseteq Y$ and $Y \subseteq X$. "Clearly",

$$\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^{2\times 1},$$

because any linear combination of vectors in $\mathbb{R}^{2\times 1}$ (with weights in the field $\mathbb{R}$) remains in $\mathbb{R}^{2\times 1}$. It remains to show that

$$\mathbb{R}^{2\times 1} \subseteq \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

Let $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^{2\times 1}$ be an arbitrary vector. We must show that there is a linear combination lying in the span that equals $\begin{bmatrix}a\\b\end{bmatrix}$. In other words, we must solve the vector equation

$$c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}.$$

Set up the augmented matrix and compute

$$\begin{bmatrix}1 & 1 & a\\0 & 1 & b\end{bmatrix} \overset{r_1^*=r_1-r_2}{\sim} \begin{bmatrix}1 & 0 & a-b\\0 & 1 & b\end{bmatrix}.$$

This shows that

$$(a-b)\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix},$$

i.e. $\begin{bmatrix}a\\b\end{bmatrix}$ may in fact be written as a linear combination of $\begin{bmatrix}1 & 0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$, proving that $\mathbb{R}^{2\times 1} \subseteq \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$, completing the proof. ■

5. (14 points) Are the following sets independent or dependent?

(a) (7 points) (Field: $\mathbb{R}$) $\left\{\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}, \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}\right\}$

Solution: We must seek a solution to the matrix equation

$$c_1\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + c_2\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix} + c_3\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Using matrix algebra, we see that the left-hand side becomes

$$\begin{bmatrix}c_1+c_2+2c_3 & 0\\c_1+2c_2+c_3 & c_1+c_3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Hence we are being asked to solve the following system of linear equations:

$$\begin{cases} c_1 & +c_2 & +2c_3 & = 0\\ 0 & & & = 0\\ c_1 & +2c_2 & +c_3 & = 0\\ c_1 & & +c_3 & = 0. \end{cases}$$

Set up an augmented matrix and compute

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \underset{\sim}{r_2 \leftrightarrow r_4} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{r_2^* = r_2 - r_1, r_3^* = r_3 - r_1} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{r_3^* = r_3 + r_2} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{r_3^* = -\frac{1}{2} r_3} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{r_2^* = r_2 + r_3, r_1^* = r_1 - 2r_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{r_1^* = r_1 - r_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This shows that the only solution to the matrix equation is $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and therefore we see that the given set of matrices is **independent**.

(b) (7 points) (Field: $\mathbb{R}$) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} \right\}$

Solution: Dependent. There are more vectors in the set than the length of each vector, so we may say it is dependent by a theorem.

6. (28 points) Perform the computation or state that it is not well-defined.

(a) (7 points) (Field: $\mathbb{R}$) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 & 4 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

Solution: Not well-defined. We can only add matrices that are of the same size.

(b) (7 points) (Field: $\mathbb{R}$) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 0 \end{bmatrix}^T$

Solution: Compute

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} + 3\begin{bmatrix}1 & 2\\3 & 0\\5 & 0\end{bmatrix}^T = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} + 3\begin{bmatrix}1 & 3 & 5\\2 & 0 & 0\end{bmatrix}$$
$$= \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} + \begin{bmatrix}3 & 9 & 15\\6 & 0 & 0\end{bmatrix}$$
$$= \begin{bmatrix}4 & 11 & 18\\10 & 5 & 6\end{bmatrix}.$$

(c) (7 points) (Field: $\mathbb{R}$) $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\1 & 1 & 0\end{bmatrix}$

Solution: Compute

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}2+2 & 1+2 & 1+0\\6+4 & 3+4 & 3+0\end{bmatrix} = \begin{bmatrix}4 & 3 & 1\\10 & 7 & 3\end{bmatrix}.$$

(d) (7 points) (Field: $\mathbb{R}$) $\begin{bmatrix}1 & 1 & 0\\2 & 6 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\0 & 2 & 5\end{bmatrix}$

Solution: Not well-defined. We may not multiply a matrix of size 2×3 on the left of a matrix of size 2×3.

Recall that the trace of a square matrix $A \in \mathbb{F}^{n\times n}$ with representation $A = \{a_{ij}\}$ is the number

$$\operatorname{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn}.$$

For instance, in $\mathbb{F} = \mathbb{R}$, $\operatorname{tr}\left(\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\right) = 1 + 5 + 9 = 15.$

7. (4 points) (Bonus) Prove that $\operatorname{tr}(A) = \operatorname{tr}(A^T)$.

Solution: Since the transpose makes the columns of A into rows and vice versa, if $A = \{a_{ij}\}$, then $A^T = \{a_{ji}\}$. In particular, observe that the diagonal is not affected by a transpose (of an $n \times n$ matrix). Therefore

$$\operatorname{tr}(A) = a_{11} + \ldots + a_{nn} = \operatorname{tr}(A^T).$$