§9.8 | **Recall**: generally, if $|x|<a$, it means $-a<x<a$

Find radius of convergence

(#6) $\sum_{n=0}^{\infty}(3x)^n$

Solution: Let $a_n=(3x)^n$ so ~~that by~~ now compute ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(3x)^{n+1}}{(3x)^n}$$

$$=\lim_{n\to\infty}|3x|$$

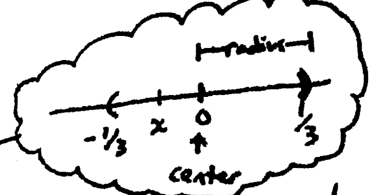
$$=|3x|$$

This limit is <1 whenever

$|3x|<1$,

or

$-\frac{1}{3}<x<\frac{1}{3}$



Therefore the radius of convergence is $\frac{1}{3}$.

(#7) $\sum_{n=1}^{\infty}\frac{(4x)^n}{n^2}$

Solution: Let $a_n=\frac{(4x)^n}{n^2}$ so compute for ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(4x)^n}\right|$$

$$=\lim_{n\to\infty}\left|4x\cdot\underbrace{\frac{n^2}{n^2+2n+1}}_{\downarrow\ 1}\right|$$

$$=|4x|$$

For convergence, we require this limit <1:

$|4x|<1$ means

$-1<4x<1$

$\Downarrow$

$-\frac{1}{4}<x<1 \Rightarrow$ R.C is $\frac{1}{4}$.

(#9) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Soln : Let $a_n = \frac{x^{2n}}{(2n)!}$. Compute for ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{2n+2}(2n)!}{(2n+2)!\,x^{2n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^2\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+1)}\right|$$

$= 0 < 1$ for all x.

Therefore the radius of convergence is ∞.

Find the interval of convergence.

(#11) $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

Soln : Let $a_n = \left(\frac{x}{4}\right)^n$. Compute ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(\frac{x}{4}\right)^{n+1} \cdot \left(\frac{4}{x}\right)^n\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{4}\right|$$

$$= \left|\frac{x}{4}\right|$$

This limit < 1 means

$$\left|\frac{x}{4}\right| < 1 \longleftrightarrow -1 < \frac{x}{4} < 1$$

$$\Downarrow$$

$$\boxed{-4 < x < 4}$$

The IoC is $(-4, 4)$.

(#14) $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

Soln: Let $a_n = (-1)^{n+1}(n+1)x^n$. Compute for ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+2}}(n+2)x^{n+1}}{\cancel{(-1)^{n+1}}(n+1)x^n}\right|$$

$$= \lim_{n\to\infty}\left|\underbrace{\frac{n+2}{n+1}}_{\downarrow\, 1} x\right|$$

$$= |x|$$

This limit is < 1 when

$$|x| < 1 \longleftrightarrow \boxed{-1 < x < 1}$$

The IoC is $(-1, 1)$.

(#19) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{6^n}$

Soln: Let $a_n = \frac{(-1)^{n+1} x^n}{6^n}$. Compute for ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+2}} x^{n+1}}{6^{n+1}} \cdot \frac{6^n}{\cancel{(-1)^{n+1}} x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{6}\right|$$

$$= \left|\frac{x}{6}\right|$$

This limit < 1 provided

$$\left|\frac{x}{6}\right| < 1 \longleftrightarrow -1 < \frac{x}{6} < 1$$

$$\Downarrow$$

$$\boxed{-6 < x < 6}$$

The IoC is $(-6, 6)$.

(#20) $\sum_{n=0}^{\infty} \frac{(-1)^n n! (x-5)^n}{3^n}$

<u>Solution</u>: Let $a_n = \frac{(-1)^n (n!)(x-5)^n}{3^n}$. Compute for ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)}^{n+1} (n+1)!(x-5)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\cancel{(-1)}^{n} n! (x-5)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(x-5)}{3} \right|$$

$$= \begin{cases} \infty \ ; & x \neq 5 \\ 0 \ ; & x = 5 \end{cases}$$

Therefore the IoC is the single point $\{5\}$,
(note: here we say RoC is 0)

(#27) $\sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$

<u>Soln</u>: Let $a_n = \frac{n}{n+1}(-2x)^{n-1}$. Compute for ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{n+1}{n+2}(-2x)^n \cdot \frac{n+1}{n(-2x)^{n-1}} \right|$$

$$= \lim_{n\to\infty} \left| \underbrace{\frac{(n+1)(n+1)}{(n+2)\,n}}_{\downarrow \, 1} (-2x) \right|$$

$$= |-2x| = |2x|$$

This limit is <1 provided

$$|2x| < 1 \longleftrightarrow -1 < 2x < 1$$
$$\Downarrow$$
$$-\tfrac{1}{2} < x < \tfrac{1}{2}$$

The IoC is $(-\frac{1}{2}, \frac{1}{2})$.

#66 (a) $J_1(x) = x\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k!\,(k+1)!}$ ⟵ Show IoC is $(-\infty,\infty)$

Soln: Let $a_k = \frac{(-1)^k x^{2k}}{2^{2k+1} k!\,(k+1)!}$. Compute for ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}\, x^{2(k+1)}}{2^{2(k+1)+1}(k+1)!\,(k+2)!}\cdot\frac{2^{2k+1}k!\,(k+1)!}{\cancel{(-1)^k}\, x^{2k}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{\cancel{x^{2k+2}}\cdot \cancel{2^{2k+1}}\cdot \cancel{k!}\cdot(k+1)\cdot\cancel{k!}}{\cancel{2^{2k+3}}(k+1)\cancel{k!}(k+2)(k+1)\cancel{k!}\,\cancel{x^{2k}}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{x^2\cdot 2\cdot(k+1)}{2^3(k+1)(k+2)(k+1)}\right|$$

$$= 0 < 1 \quad (\text{for } \underline{\underline{\text{all}}}\ x)$$

Therefore the IoC is $(-\infty,\infty) = \mathbb{R}$.

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§9.9 Find a power series for the function centered at c and determine the IoC.

(#6) $f(x) = \frac{2}{6-x}$; $c = -2$

Soln: Since $c=-2$, we must make

$$(x-c) = (x-(-2)) = (x+2)$$

appear. So,

$$f(x) = \frac{2}{6-(x+\underbrace{2-2}_{})} = 2\cdot\frac{1}{(6+2)-(x+2)} = 2\cdot\frac{1}{8-(x+2)}$$

"add convenient form of zero"

$$= 2\cdot\frac{1}{8-(x+2)}\left(\frac{1/8}{1/8}\right)$$

"multiply by convenient form of 1"

$$= \frac{2}{8}\cdot\frac{1}{1-\left(\frac{x+2}{8}\right)}$$

By geometric series, we now see that

$$f(x) = \frac{2}{8}\sum_{k=0}^{\infty}\left(\frac{x+2}{8}\right)^k = \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{8^k}(x+2)^k$$

which converges whenever

$$-1 < \frac{x+2}{8} < 1 \iff -8 < x+2 < 8 \longleftrightarrow -10 < x < 6,$$

So IoC is $(-10, 6)$.

(#10) $f(x) = \dfrac{3}{3x+4}$; $c=2$

<u>Soln</u>: Since $c=2$, we must make

$$(x-c) = (x-2)$$

appear. So,

$$f(x) = \frac{3}{3x+4} = \frac{3}{3(\underbrace{x-2+2}_{\text{"add convenient form of 0"}})+4} = 3 \cdot \frac{1}{3(x-2)+6+4}$$

$$= 3 \cdot \frac{1}{3(x-2)+10}$$

$$= 3 \cdot \frac{1}{10-(-3(x-2))}$$

$$= 3 \cdot \frac{1}{10-(-3(x-2))} \; \frac{1/10}{1/10} \quad \Big\} \text{ multiply by convenient form of 1}$$

$$= \frac{3}{10} \cdot \frac{1}{1-\left(-\frac{3}{10}(x-2)\right)}$$

By geometric series, we now see that

$$f(x) = \frac{3}{10} \; \frac{1}{1-\left(-\frac{3}{10}(x-2)\right)}$$

$$= \frac{3}{10} \sum_{k=0}^{\infty} \left(-\frac{3}{10}(x-2)\right)^k = \sum_{k=0}^{\infty} \left(-\frac{3}{10}\right)^k (x-2)^k$$

which converges whenever

$$-1 < -\frac{3}{10}(x-2) < 1 \longleftrightarrow \frac{10}{3} > x-2 > -\frac{10}{3}$$

$$\longleftrightarrow \frac{10}{3}+2 > x > -\frac{10}{3}+2$$

$$\longleftrightarrow \frac{10}{3}+\frac{6}{3} > x > -\frac{10}{3}+\frac{6}{3}$$

$$\longleftrightarrow \frac{16}{3} > x > -\frac{4}{3}$$

Thus IoC is $\left(-\frac{4}{3}, \frac{16}{3}\right)$.

Use power series $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ to determine a power series, centered at zero, for the given function. Find IoC.

#19 $f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$

Soln: First note that

$$\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n, \longleftarrow \text{valid for } -1<x<1$$

so

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left(\frac{1}{x+1}\right)$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n x^n\right]$$

$$= \sum_{n=1}^{\infty} n(-1)^n x^n$$

What is IoC of this?

Let $a_n = n(-1)^n x^n$, compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right|$

$= |x|$

which is <1 provided

$|x|<1 \longleftrightarrow -1<x<1$

Therefore the IoC is $(-1,1)$.

#21 $f(x) = \ln(x+1) = \int \frac{1}{x+1}\,dx$

Soln: Since $\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n, \longleftarrow$ valid for $-1<x<1$

compute

$$f(x) = \ln(x+1) = \int \frac{1}{1+x}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \,dx$$

integral of sum is sum of integrals $\longrightarrow$

$$= \sum_{n=0}^{\infty} (-1)^n\left(\int x^n\,dx\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

This series also has IoC $(-1,1)$ (by ratio test).

#36 Find series representation of

$$f(x) = \frac{x}{(1-x)^2}$$

Soln: By geometric series,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \leftarrow \text{ valid for } -1<x<1$$

$\downarrow$ differentiate both sides

$$\frac{-1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k$$

$\downarrow$ multiply by -1

$$\frac{1}{(1-x)^2} = -\sum_{k=1}^{\infty} kx^k$$

Therefore,

$$f(x) = x \cdot \frac{1}{(1-x)^2} = x \cdot \left(-\sum_{k=1}^{\infty} kx^k\right)$$

$$= -\sum_{k=1}^{\infty} kx^{k+1}$$

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§9.10 Use definition of Taylor series to find the Taylor series of the given function centered at c.

(#7) $f(x) = \ln(x)$; $c = 1$

Soln:

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{2}{x^3}$$

$$f^{(4)}(x) = -\frac{2\cdot 3}{x^4}$$

$$\vdots$$

$$f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{x^k}$$

$\rightarrow$ Therefore,

$$\frac{f^{(k)}(c)}{k!} = \frac{f^{(k)}(1)}{k!} = \frac{(-1)^{k+1}(k-1)!}{1^k k!}$$

$$= \frac{(-1)^{k+1}(k-1)!}{k!}$$

$$= \frac{(-1)^{k+1}}{k}$$

Hence, by Taylor series formula,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

#8 $f(x)=e^x;\ c=1$

Soln: $f'(x)=e^x$

$f''(x)=e^x$

$\vdots$

$f^{(k)}(x)=e^x$

Therefore,

$$\frac{f^{(k)}(c)}{k!}=\frac{f^{(k)}(1)}{k!}=\frac{e}{k!}$$

Hence, by Taylor series formula,

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$=\sum_{k=0}^{\infty}\frac{e}{k!}(x-1)^k$$

Find MacLaurin series (i.e. Taylor series centered at $c=0$) of given function. Use table on pg. 670.

#28 $g(x)=e^{-3x}$

Soln: From table,

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Therefore, substitute in $-3x$ for x to get

$$e^{-3x}=\sum_{k=0}^{\infty}\frac{(-3x)^k}{k!}=\sum_{k=0}^{\infty}\frac{(-3)^k}{k!}x^k$$

#30 $f(x)=\ln(1+x^2)$

Soln: From table,

$$\ln(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(x-1)^k}{k}$$

Therefore substitute in $1+x^2$ for x to get

$$\ln(1+x^2)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(1+x^2-1)^k}{k}$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{2k}}{k}$$

#31 $g(x) = \sin(3x)$

Soln: From table,

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Therefore substitute in $3x$ for x to get

$$\sin(3x) = \sum_{k=0}^{\infty} \frac{(-1)^k (3x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{(2k+1)!} x^{2k+1}$$

#53 Find Maclaurin series for

$$f(x) = \int_0^x e^{-t^2} - 1 \, dt$$

Soln: Start with

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$\downarrow$ plug in $x = -t^2$

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}$$

notice that the $k=0$ term of the sum equals 1

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{k!}$$

Thus

$$e^{-t^2} - 1 = \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{k!}$$

Now integrate:

$$\int_0^x e^{-t^2} - 1 \, dt = \int_0^x \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{k!} \, dt$$

integral of sum is sum of integrals

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(\int_0^x t^{2k} \, dt \right)$$

this integral is:

$$\int_0^x t^{2k} dt = \frac{t^{2k+1}}{2k+1} \Big|_0^x = \frac{x^{2k+1}}{2k+1} - 0 = \frac{x^{2k+1}}{2k+1}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k+1}}{2k+1}$$