$\S 9.4$ #4 $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converges because

$$3n^2+2 > 3n^2$$

$\Downarrow$ (reciprocal)

$$\frac{1}{3n^2+2} < \frac{1}{3n^2}$$

and we know that

$$\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges by p-series test $\left(=\frac{1}{3}\left(\frac{\pi^2}{6}\right)\right)$

"Basel problem"

So direct comparison test says $\sum_{n=1}^{\infty} \frac{1}{3n^2+2} < \sum_{n=1}^{\infty} \frac{1}{3n^2} < \infty$

converges!

$\S 9.4$ #5 $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges because

$\sqrt{n} < n$ and $n < n+1$

subtr. 1 $\Downarrow$ $\Downarrow$

$\sqrt{n}-1 < n-1$ // $n-1 < n$

$\downarrow$ reciprocal

$$\frac{1}{\sqrt{n}-1} > \frac{1}{n-1} \quad \text{and} \quad \frac{1}{n-1} > \frac{1}{n}$$

But since

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series!!),

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{n-1} > \sum_{n=2}^{\infty} \frac{1}{n} = \infty,$$

we see by direct comparison test that

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges

$\boxed{\S 9.4 \ \#14}$ $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n+1}$ converges because

if we let $a_n = \dfrac{5}{4^n+1}$ and $b_n = \dfrac{5}{4^n}$, we see that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{5}{4^n+1}}{\frac{5}{4^n}} \right)$$

$$= \lim_{n\to\infty} \frac{\cancel{5}}{4^n+1} \cdot \frac{4^n}{\cancel{5}}$$

$$= \underline{1} > 0$$

finite

and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{4^n} = 5 \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \left(= 5\left(\frac{1/4}{1-1/4}\right) = \frac{5}{3} \right)$$

converges... geometric

so by limit comparison test,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5}{4^n+1} \text{ also converges}$$

§9.4 # 15 $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges because if we let

$a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$, we see

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}}\right)(n)$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1+\underbrace{1/n^2}_{\to 0}}}$$

$$= \sqrt{1} = 1 > 0$$

finite

~~so~~ and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic series)}$$

therefore by limit comparison test,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}} \text{ also diverges}$$

§9.5 #7 $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ converges because if $a_n = \frac{1}{3^n}$,

then

① $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3^n} = 0$, and

② $a_{n+1} = \frac{1}{3^{n+1}} = \frac{1}{3}\frac{1}{3^n} = \frac{1}{3}a_n < a_n$

hold, so by alternating series test,

$$\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} \text{ converges } \left(= \frac{-1/3}{1-(-1/3)} \right)$$

geometric

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§9.5 #9 $\sum_{n=1}^{\infty} \frac{(-1)^n(5n-1)}{4n+1}$ diverges because if

$a_n = \frac{(-1)^n(5n-1)}{4n+1}$, then $\lim_{n\to\infty} a_n$ DNE

(why? Because $\frac{5n-1}{4n+1} \to \frac{5}{4}$ while $(-1)^n$ oscillates forever)

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§9.5 #10 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$ converges because if $a_n = \frac{n}{n^2+5}$,

then

① $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+5}$ 0, and

② $a_{n+1} = \frac{n+1}{(n+1)^2+5} \leq a_n$ (must do this... positive sequence that converges to zero!)

hold, so by alternating series test,

$$-\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges,}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$$

§9.5 #19 $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges because if $a_n = \frac{1}{n!}$, then

① $\lim_{n\to\infty} \frac{1}{n!} = 0 = \lim_{n\to\infty} a_n$, and

② $a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$

hold, so by alternating series test,

$$\sum_{n=0}^{\infty} (-1)^n a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \text{ converges } \left(= \frac{1}{e}\right)$$

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§9.6 #14 $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because if $a_n = \frac{1}{n!}$,

then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$$
$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$
$$= \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$
$$= \lim_{n\to\infty} \frac{1}{n}$$
$$= 0 < 1,$$

so by ratio test,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges } (= e)$$

§9.6 #15 $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$ diverges because if $a_n = \frac{n!}{3^n}$,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{3} = \infty \text{ (diverge)}$$

$$> 1$$

Therefore by ratio test,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n!}{3^n} \text{ diverges}$$

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§9.6 #36 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$ converges because if $a_k = \frac{1}{k^k}$, then

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\frac{1}{k^k}}$$

$$= \lim_{k\to\infty} \frac{1}{k}$$

$$= 0 < 1.$$

Therefore by root test,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^k} \text{ converges.}$$

§9.6 #37

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \frac{1}{2} < 1$$

Therefore by root test,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n \text{ converges}$$
