§9.2 #7

$\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$ diverges because it is a geometric series, but with $r = \frac{7}{6} > 1$, so it diverges by Theorem 9.6

§9.2 #18

$\sum_{n=0}^{\infty} (-0.6)^n$ converges because it is geometric with $-1 < (r = -0.6) < 1$, hence it converges to the value

$$\sum_{n=0}^{\infty} (-0.6)^n = \frac{1}{1-(-0.6)} = \frac{1}{1.06} \approx 0.9433$$

§9.2 #25

$$\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n = 5\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 5\left(\frac{1}{1-2/3}\right)$$

geo. series

$$= 5\left(\frac{1}{1/3}\right)$$

$$= 15$$

§9.2 #28

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{(2n+1)(2n+3)}}_{\text{partial frac}} =$$

$$\frac{1}{(2n+1)(2n+3)} = \frac{A}{2n+1} + \frac{B}{2n+3} = \frac{1}{4n+2} - \frac{1}{4n+6} \leftarrow \text{telescoping}$$

$$1 = 2nA + 3A + 2nB + B$$

$$1 = 2n(A+B) + (3A+B)$$

$$\Rightarrow \begin{cases} 2A + 2B = 0 \rightarrow A = -B \\ 3A + B = 1 \end{cases}$$

$$-3B + B = 1$$

$$\boxed{B = -\frac{1}{2}} \rightarrow \boxed{A = \frac{1}{2}}$$

§9.2 #28 cont

$4(k+1)+2 = 4k+6$

$4(k+1)+6 = 4k+10$

$$S_n = \sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)} = \sum_{k=1}^{n}\left(\frac{1}{4k+2} - \frac{1}{4k+6}\right)$$

partial sum

$$= \left(\frac{1}{6} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{10}} - \cancel{\frac{1}{14}}\right) + \left(\cancel{\frac{1}{14}} - \frac{1}{18}\right) + \dots +$$

$$+ \left(\cancel{\frac{1}{4(n-1)+2}} - \underbrace{\frac{1}{4(n-1)+6}}_{-\cancel{\frac{1}{4n+2}}}\right) + \left(\cancel{\frac{1}{4n+2}} - \frac{1}{4n+6}\right)$$

$$= \frac{1}{6} - \frac{1}{4n+6}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left(\frac{1}{6} - \frac{1}{4n+6}\right)$$

$\frac{1}{4n+6} \to 0$

$$= \frac{1}{6}$$

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§9.2 #41

$\sum_{k=0}^{\infty} (1.075)^k$ diverges because it is geometric with $r = 1.075 > 1$

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§9.2 #44

$\sum_{k=1}^{\infty} \frac{4k+1}{3k-1}$ diverges because if $a_k = \frac{4k+1}{3k-1}$,

then $\lim_{k\to\infty} a_k = \frac{4}{3} \neq 0$, therefore by Theorem 9.9,

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{4k+1}{3k-1}$ diverges

§9.2 #46 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k+1} - \frac{1}{k+2}$ is telescoping, so

compute the partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k+1} - \frac{1}{k+2}$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2} - \frac{1}{n+2}$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{k+1} - \frac{1}{k+2} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+2} \to 0$$

$$= \frac{1}{2}$$

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§9.2 #61 $\displaystyle\sum_{k=1}^{\infty} (3x)^k$ converges, as geometric series,

whenever $-1 < 3x < 1$

$\downarrow$ div by 3

$$\boxed{-\frac{1}{3} < x < \frac{1}{3}}$$

$\boxed{\S 9.3\ \#2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+5}$ diverges because

$$\int_1^{\infty} \frac{2}{3x+5}\,dx \underset{\uparrow}{=} \frac{2}{3}\int_8^{\infty} \frac{1}{u}\,du = \frac{2}{3} \lim_{b\to\infty} \ln(u)\Big|_8^b$$

$u = 3x+5$

$\frac{1}{3}du = dx$

$$= \frac{2}{3} \lim_{b\to\infty} \big(\underbrace{\ln(b)}_{\to\infty} - \ln(8)\big)$$

$$= \infty,$$

therefore

$$\int_1^{\infty} \frac{2}{3x+5}\,dx \text{ diverges, and so}$$

by the integral test,

$$\sum_{k=1}^{\infty} \frac{2}{3k+5} \text{ also diverges.}$$

$\boxed{\S 9.3\ \#5}$ $\displaystyle\sum_{n=1}^{\infty} e^{-n}$ converges because

$$\int_1^{\infty} e^{-x}\,dx \underset{\uparrow}{=} -\int_{-1}^{-\infty} e^{u}\,du = \int_{-\infty}^{-1} e^{u}\,du$$

$u = -x$

$du = -dx$

$$= \lim_{a\to-\infty} \int_a^{-1} e^{u}\,du$$

$$= \lim_{a\to-\infty} e^{u}\Big|_a^{-1}$$

$$= \lim_{a\to-\infty} e^{-1} - \underbrace{e^{a}}_{\to 0}$$

$\lim_{x\to-\infty} e^{x} = 0$

$$= e^{-1},$$

therefore

$$\int_1^{\infty} e^{-x}\,dx \quad \text{converges}$$

and by integral test,

$$\sum_{n=1}^{\infty} e^{-n} \quad \text{converges}$$

$\boxed{\S 9.3 \ \# 33}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ diverges because

$\frac{1}{\sqrt[5]{n}} = \frac{1}{n^{1/5}}$, and so, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

is a p-series with $0 < (p = \frac{1}{5}) \leq 1$

and hence by Theorem 9.11, diverges

$\boxed{\S 9.3 \ \# 38}$ $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ converges because

it is a p-series with $(p = \pi = 3.14\ldots) > 1$

and so by Theorem 9.11, converges