MATH 3315 - EXAM 3 - FALL 2017
SOLUTION

Friday 27 October 2017
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\int \log(x)\mathrm{d}x = x\log(x) - x + C$$

To integrate $\sin^m(x)\cos^n(x)$...

m odd	$\longrightarrow$ save a sine and convert rest to cosines
n odd	$\longrightarrow$ save a cosine and convert rest to sines both m and n even
m even and n even	$\longrightarrow$ convert all to cosines

The trigonometric substitutions are...

$\sqrt{a^2-x^2}$	$\longrightarrow x = a\sin(\theta)$
$\sqrt{a^2+x^2}$	$\longrightarrow x = a\tan(\theta)$
$\sqrt{x^2-a^2}$	$\longrightarrow x = a\sec(\theta)$

1. (50 points) Find the antiderivative.

(a) (10 points) $\int xe^{-18x}\mathrm{d}x$

Solution: Let $u = x$ and $\mathrm{d}v = e^{-18x}$, then $\mathrm{d}u = \mathrm{d}x$ and $v = -\frac{1}{18}e^{-18x}$. By integration by parts, compute

$$\begin{aligned}\int xe^{-18x}\mathrm{d}x &= -xe^{-18x} + \frac{1}{18}\int e^{-18x}\mathrm{d}x \\ &= -xe^{-18x} - \frac{1}{18^2}e^{-18x} + C \\ &= -xe^{-18x} - \frac{1}{324}e^{-18x} + C.\end{aligned}$$

(b) (10 points) $\int \frac{x}{x^3 - 4x}\mathrm{d}x$

Solution: Factor the denominator:

$$x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2).$$

Therefore we will use partial fractions to decompose: write

$$\frac{x}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}.$$

Multiply by the common denominator $x(x-2)(x+2)$ to both sides to get

$$x = A(x^2 - 4) + B(x^2 + 2x) + C(x^2 - 2x) = (A + B + C)x^2 + (2B - 2C)x + (-4A).$$

Equating coefficients leads to the following system of equations:

$$\begin{cases} A \quad +B \quad +C &= 0 \quad (i) \\ \qquad 2B \quad -2C &= 1 \quad (ii) \\ -4A &= 0 \quad (iii) \end{cases}$$

From (iii), we deduce $A = 0$. Plugging this into (i) and solving for B yields $B = -C$. Plugging this into (ii) yields $-2C - 2C = 1$, and hence $-4C = 1$, so $C = -\frac{1}{4}$. Hence $B = -C = \frac{1}{4}$. Thus we have shown

$$\frac{x}{x^3 - 4x} = \frac{1}{4}\frac{1}{x-2} - \frac{1}{4}\frac{1}{x+2}.$$

Therefore we may compute

$$\begin{aligned}\int \frac{x}{x^3 - 4x}\mathrm{d}x &= \frac{1}{4}\int \frac{1}{x-2}\mathrm{d}x - \frac{1}{4}\int \frac{1}{x+2}\mathrm{d}x \\ &= \frac{1}{4}\log(x-2) - \frac{1}{4}\log(x+2) + C.\end{aligned}$$

(c) (10 points) $\int \sin^2(x)\mathrm{d}x$

Solution: Apply the reduction formula $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ and compute

$$\begin{aligned}\int \sin^2(x)\mathrm{d}x &= \int \frac{1}{2}\mathrm{d}x - \frac{1}{2}\int \cos(2x)\mathrm{d}x \\ &= \frac{x}{2} - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

(d) (10 points) $\int e^x \sin(x)\mathrm{d}x$

Solution: We will apply integration by parts twice. Let $u = e^x$ and $\mathrm{d}v = \sin(x)$. Then $\mathrm{d}u = e^x\mathrm{d}x$ and $v = -\cos(x)$. Therefore integration by parts yields

$$(*) \qquad \int e^x \sin(x)\mathrm{d}x = -e^x\cos(x) + \int e^x \cos(x)\mathrm{d}x$$

We will now apply integration by parts to the integral we got. This time let $u_2 = e^x$ and $\mathrm{d}v_2 = \cos(x)$. Then $\mathrm{d}u_2 = e^x\mathrm{d}x$ and $v_2 = \sin(x)$. Hence we compute

$$(**) \qquad \int e^x \cos(x)\mathrm{d}x = e^x\sin(x) - \int e^x \sin(x)\mathrm{d}x.$$

Combining $(*)$ and $(**)$ yields the equation

$$\int e^x \sin(x)\mathrm{d}x = -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x)\mathrm{d}x.$$

Adding $\int e^x \sin(x)\mathrm{d}x$ to both sides and then dividing by 2 finally yields

$$\int e^x \sin(x)\mathrm{d}x = \frac{e^x(\sin(x) - \cos(x))}{2}.$$

(e) (10 points) $\int x\sqrt{9 - x^2}\mathrm{d}x$

Solution: This integral could be solved two ways: via a trig substitution or via u-substitution. We will solve it both ways:

via u-substitution

Let $u = 9 - x^2$ so $-\frac{1}{2}\mathrm{d}u = x\mathrm{d}x$. Then we compute

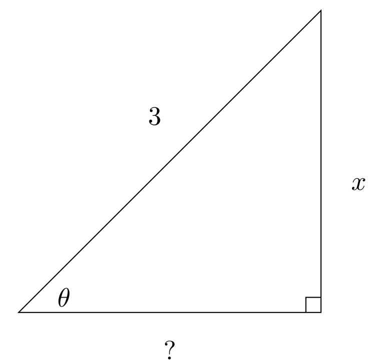
$$\begin{aligned}\int x\sqrt{9-x^2}\mathrm{d}x &= -\frac{1}{2}\int u^{\frac{1}{2}}\mathrm{d}u \\ &= -\frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= -\frac{1}{3}(9-x^2)^{\frac{3}{2}} + C.\end{aligned}$$

via trig substitution

Let $x = 3\sin(\theta)$ then $\mathrm{d}x = 3\cos(\theta)\mathrm{d}\theta$. Substitute these values into the integral to compute

$$\begin{aligned}\int x\sqrt{9-x^2}\mathrm{d}x &= \int (3\sin(\theta))\sqrt{9 - 9\sin^2(\theta)}(3\cos(\theta))\mathrm{d}\theta \\ &= 27\int \sin(\theta)\cos(\theta)\sqrt{1-\sin^2(\theta)}\mathrm{d}\theta \\ &= 27\int \sin(\theta)\cos^2(\theta)\mathrm{d}\theta \\ &\overset{u=\cos(\theta)}{=} -27\int u^2\mathrm{d}u \\ &= -9u^3 + C \\ &= -9\cos^3(\theta) + C.\end{aligned}$$

Now we find the value of $\cos(\theta)$. Draw a triangle matching $\frac{x}{3} = \sin(\theta)$:



Find ? using the Pythagorean theorem: $?^2 + x^2 = 3^2$, so $? = \sqrt{9 - x^2}$. Therefore $\cos(\theta) = \dfrac{\sqrt{9-x^2}}{3}$. Finally we may compute

$$\int x\sqrt{9-x^2}\mathrm{d}x = -9\cos^3(\theta) + C = -9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C = -\frac{1}{3}(9-x^2)^{\frac{3}{2}} + C.$$

2. (30 points) Compute the following improper integrals or show that they do not converge.

(a) (10 points) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\mathrm{d}x$

Solution: We compute

$$\begin{aligned}
\int_2^\infty \frac{1}{\sqrt{x}}\mathrm{d}x &= \lim_{b\to\infty}\int_2^b x^{-\frac{1}{2}}\mathrm{d}x \\
&= \lim_{b\to\infty} \left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_2^b \\
&= 2\lim_{b\to\infty}\sqrt{b} - \sqrt{2} \\
&= \infty,
\end{aligned}$$

so the integral diverges.

(b) (10 points) $\displaystyle\int_0^\infty e^{-2x+3}\mathrm{d}x$

Solution: We will use the u-substitution $u = -2x + 3$ (so $-\frac{1}{2}\mathrm{d}u = \mathrm{d}x$) to compute

$$\begin{aligned}
\int_0^\infty e^{-2x+3}\mathrm{d}x &= \lim_{b\to\infty}\int_0^b e^{-2x+3}\mathrm{d}x \\
&= -\frac{1}{2}\lim_{b\to\infty}\int_3^{-b} e^u\mathrm{d}u \\
&= -\frac{1}{2}\lim_{b\to\infty} e^u\Big|_3^{-b} \\
&= -\frac{1}{2}\lim_{b\to\infty}\left(e^{-b} - e^3\right) \\
&= -\frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{e^b} - e^3\right) \\
&= \frac{e^3}{2}.
\end{aligned}$$

(c) (10 points) $\int_0^{\pi} \log(x^3)\mathrm{d}x$

Solution: First compute

$$\begin{aligned}\int_0^{\pi} \log(x^3)\mathrm{d}x &= 3 \lim_{a\to 0^+} \int_a^{\pi} \log(x)\mathrm{d}x \\ &= 3 \lim_{a\to 0^+} x\log(x) - x\Big|_a^{\pi} \\ &= 3 \lim_{a\to 0^+} [(a\log(a) - a) - (\pi\log(\pi) - \pi)]\end{aligned}$$

To evaluate the limit, it is sufficient to find the limit of $a\log(a)$. Since $\lim_{a\to 0^+} a\log(a) = 0\cdot(-\infty)$ is an indeterminant form, we will use L'Hopital's rule to compute

$$\lim_{a\to 0^+} a\log(a) = \lim_{a\to 0^+} \frac{\log(a)}{\frac{1}{a}} \overset{\text{L.H.}}{=} \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0^+} -a = 0.$$

Thus we may compute

$$\int_0^{\pi} \log(x^3)\mathrm{d}x = 3\lim_{a\to 0^+} [(a\log(a) - a) - (\pi\log(\pi) - \pi)] = 3(\pi\log(\pi) - \pi).$$

3. (20 points) Compute the limit or show that it does not exist.

(a) (10 points) $\lim_{x\to 1} \frac{x\sin(x^2-1)}{x^2-1}$

Solution: Plugging in $x = 1$ yields a $\frac{0}{0}$ indeterminant form. Therefore we will use L'Hopital's rule to compute

$$\begin{aligned}\lim_{x\to 1} \frac{x\sin(x^2-1)}{x^2-1} &= \lim_{x\to 1} \frac{\sin(x^2-1) + 2x^2\cos(x^2-1)}{2x} \\ &= \frac{0 + 2(1^2)(1)}{2} \\ &= 1\end{aligned}$$

(b) (10 points) $\lim_{x\to\infty} \frac{e^{x^2+1}}{1+e^{3x^2+2x+5}}$

Solution: Plugging in $x = \infty$ yields an indeterminant form $\frac{\infty}{\infty}$. Therefore we use L'Hopital's rule to compute

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{x^2+1}}{1+e^{3x^2+2x+5}} &\overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{(2x+1)e^{x^2+1}}{(6x+2)e^{3x^2+2x+5}} \\ &= \lim_{x\to\infty} \frac{2x+1}{6x+2}\frac{1}{e^{2x^2+2x+4}} \\ &= 0,\end{aligned}$$

since $\frac{2x+1}{6x+2}$ approaches $\frac{2}{6}$ as $x\to\infty$ and the exponential drags the limit down to zero.

4. (1000 points) (**BONUS**) (don't try this one until you are done with everything else!) Compute

$$\int_0^{\frac{\pi}{2}} x\log(\sin(x))\mathrm{d}x$$

Solution: The value of this integral is related to Apery's constant and finding its value would make you a famous person!