MATH 3315 - EXAM 2 - FALL 2017
SOLUTION

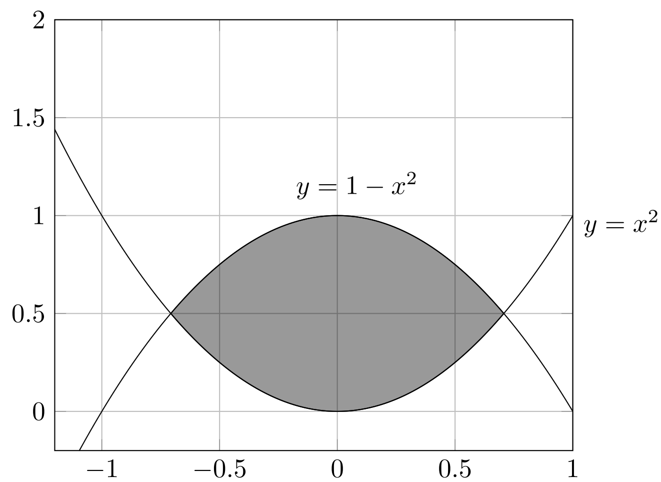
Friday 29 September 2017
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (14 points) Consider the region bounded by the curves $y = x^2$ and $y = 1 - x^2$.

   (a) (6 points) Draw this region.
   Solution: Draw



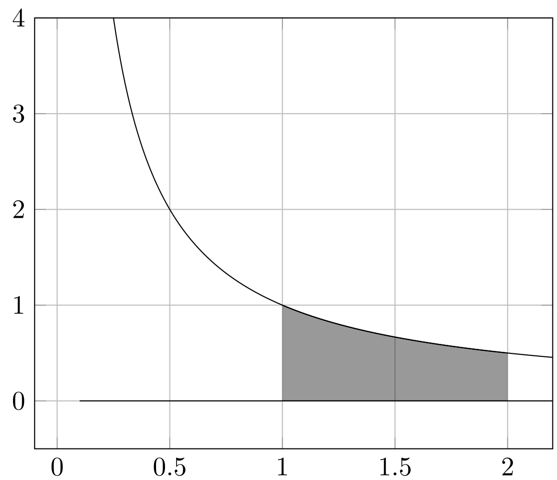
   (b) (8 points) Find the area of this region in any way you wish.

   Solution: The intersection points occur whenever $x^2 = 1 - x^2$, which has solution $x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$. We will compute it using a dx integral. Compute

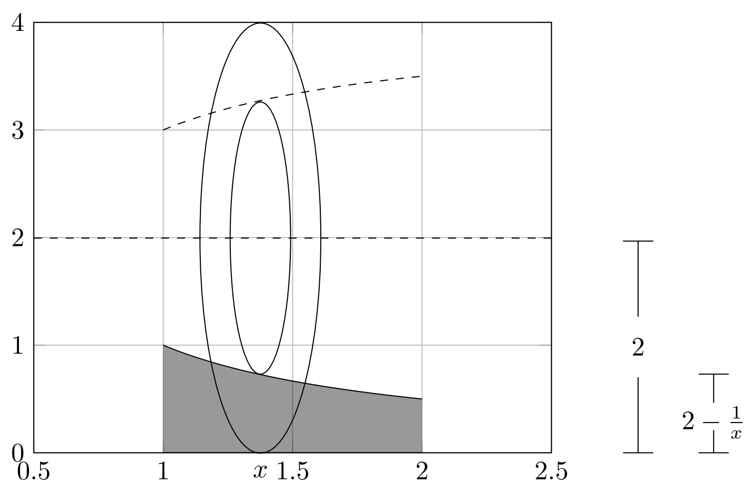
$$\begin{aligned}
\text{Area} &= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \text{top}(x) - \text{bottom}(x)\mathrm{d}x \\
&= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} (1 - x^2) - x^2\mathrm{d}x \\
&= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 1 - 2x^2\mathrm{d}x \\
&= x - \frac{2}{3}x^3\Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \\
&= \left[\frac{\sqrt{2}}{2} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] - \left[-\frac{\sqrt{2}}{2} - \frac{2}{3}\left(-\frac{\sqrt{2}}{2}\right)^3\right] \\
&= \sqrt{2} - \frac{\sqrt{2}}{3} \\
&= \frac{2\sqrt{2}}{3}.
\end{aligned}$$

2. (15 points) Consider the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$, and $x = 2$ rotated about the line $y = 2$.

   (a) (7 points) Draw the region.
   Solution: Draw



(b) (8 points) Find the volume of the region in any way you wish.
Solution: We use the washer method:



Now compute the volume

$$\begin{aligned}
\text{Volume} &= \pi \int_1^2 2^2 - \left(2 - \frac{1}{x}\right)^2 \mathrm{d}x \\
&= \pi \int_1^2 4 - \left(4 - \frac{4}{x} + \frac{1}{x^2}\right) \mathrm{d}x \\
&= \pi \int_1^2 \frac{4}{x} - \frac{1}{x^2} \mathrm{d}x \\
&= \pi \left[4\ln(x) + \frac{1}{x} \right|_1^2 \\
&= \pi \left[\left(4\ln(2) + \frac{1}{2}\right) - \left(4\ln(1) + \frac{1}{1}\right) \right] \\
&= 4\pi \ln(2) + \frac{\pi}{2} - \pi \\
&= 4\pi \ln(2) - \frac{\pi}{2}.
\end{aligned}$$

3. (8 points) (a) (4 points) Convert the point $\left(3, \frac{\pi}{3}\right)$ in polar coordinates into rectangular coordinates.

Solution: We are given $r = 3$ and $\theta = \frac{\pi}{3}$. We compute

$$x = r\cos(\theta) = 3\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

and we compute

$$y = r\sin(\theta) = 3\sin\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}.$$

Therefore the rectangular coordinates are $(x, y) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

(b) (4 points) Convert the point $(5, -5)$ in rectangular coordinates into polar coordinates.

Solution: Compute

$$r = \sqrt{x^2 + y^2} = \sqrt{5^2 + (-5)^2} = \sqrt{50}.$$

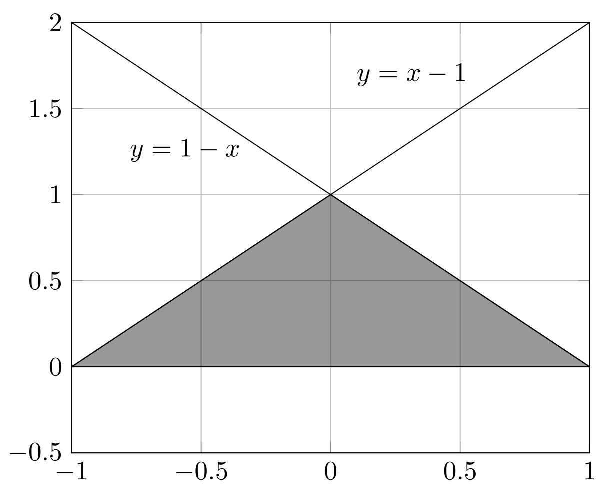
Since $(5, -5)$ is in quadrant IV, we compute

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-5}{5}\right) = \arctan(-1) = -\frac{\pi}{4}.$$

4. (20 points) Consider the region bounded by the curves $y = 1 - x$, $y = x + 1$, and the line $y = 0$.

(a) (6 points) Draw this region.

Solution: Draw:



(b) (7 points) Set up **but do not evaluate** dx integral(s) that compute the area of this region.

Solution: Write

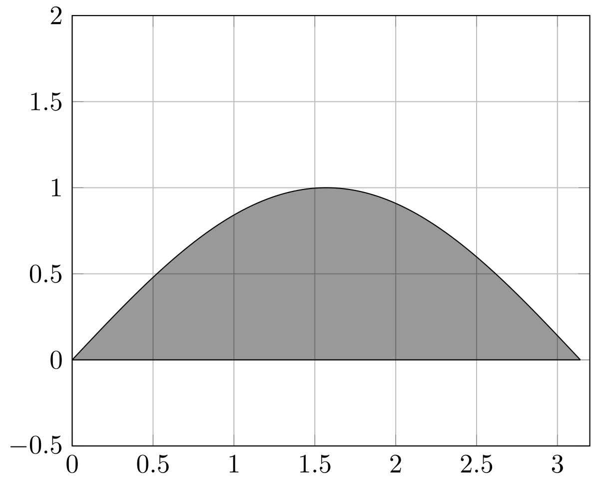
$$\text{Area} = \int_{-1}^{0} (x+1) - 0\mathrm{d}x + \int_{0}^{1} (1-x) - 0\mathrm{d}x = \int_{-1}^{0} x + 1\mathrm{d}x + \int_{0}^{1} 1 - x\mathrm{d}x$$

(c) (7 points) Set up **but do not evaluate** dy integral(s) that compute the area of this region.

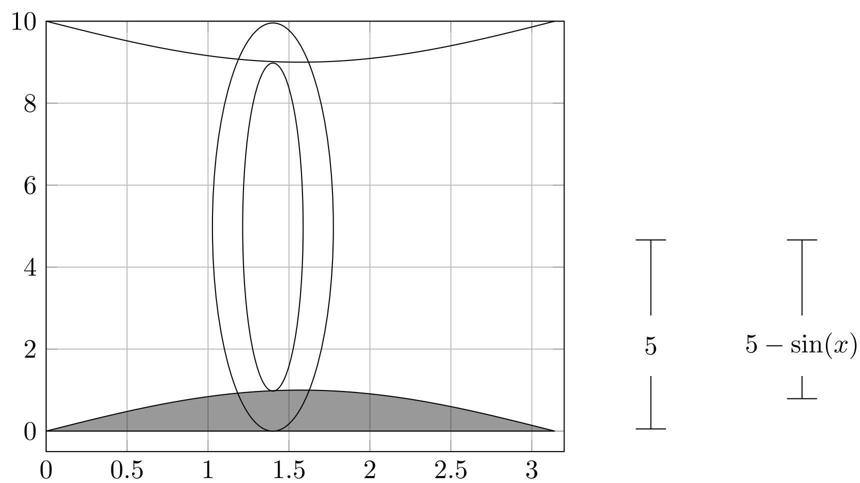
Solution: Rewrite $y = 1 - x$ as $x = 1 - y$ and rewrite $y = x + 1$ as $x = y - 1$. So compute

$$\text{Area} = \int_{0}^{1} (1-y) - (y-1)\mathrm{d}y = \int_{0}^{1} 2 - 2y\mathrm{d}y.$$

5. (13 points) Consider the region bounded by $y = \sin(x)$, $x = 0$, $x = \pi$, and $y = 0$ about the line $y = 5$.
Solution: Draw the region:



(a) (6 points) Draw the region and draw a washer.
Solution: Now draw a washer



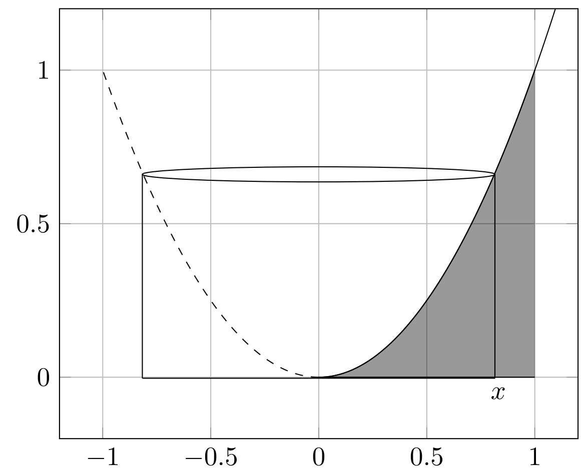
(b) (7 points) Use the washer method to find **but do not evaluate** an integral that computes the volume.
Solution: Compute

$$\text{Volume} = \pi \int_0^{\pi} 5^2 - (5 - \sin(x))^2 \mathrm{d}x.$$

6. (13 points) Consider the region bounded by $y = x^2$ and $x = 1$ rotated about the y-axis.

(a) (6 points) Draw the region and draw a shell.

Solution:



(b) (7 points) Use the shell method find **but do not evalute** an integral that computes the volume.

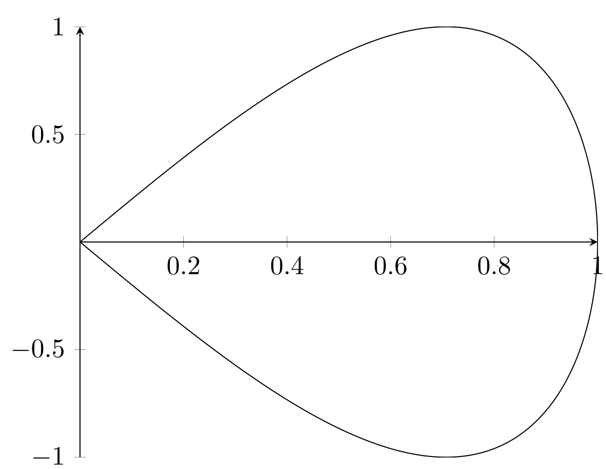
Solution: The radius of this shell at x is is x and the height is x^2. Therefore the volume is given by

$$\text{Volume} = 2\pi \int_0^1 x(x^2)\mathrm{d} = 2\pi \int_0^1 x^3 \mathrm{d}x.$$

7. (17 points) The plot of the parametric equations

$$\begin{cases} x(t) &= \sin(t) \\ y(t) &= \sin(2t) \end{cases}$$

for $0 \le t \le \pi$ is shown:



(a) (5 points) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution: We know that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)} = \frac{2\cos(2t)\sin(2t)}{\cos(t)}.$$

(b) (6 points) The t-value $t = \frac{\pi}{4}$ describes the point $(x, y) = \left(\frac{\sqrt{2}}{2}, 1\right)$. Find an equation for the tangent line at this point.
Solution: First find the slope of this tangent line:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{\pi}{4}} = \frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{4}\right)} = 0.$$

Thus the tangent line to the curve at this point is $y - 1 = 0\left(x - \frac{\sqrt{2}}{2}\right)$, i.e., $y = 1$.

(c) (6 points) Find an equation for the tangent line at the point defined by the t-value $t = \frac{\pi}{6}$.
Solution: First find the slope:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{\pi}{6}} = \frac{2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{6}\right)} = 1.$$

Now find the point corresponding to the t-value $t = \frac{\pi}{6}$:

$$x\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2},$$

and

$$y\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Therefore the tangent line we seek is the line going through $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ with slope 1, i.e.

$$y - \frac{\sqrt{3}}{2} = 1\left(x - \frac{1}{2}\right).$$