MATH 3315 - EXAM 1 - FALL 2017
SOLUTION

Friday 8 September 2017
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (36 points) Compute the following derivatives.

   (a) (6 points) $\frac{\mathrm{d}}{\mathrm{d}x}\log(5x^2+3x+1)$.

   Solution: Compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\log(5x^2+3x+1) &= \frac{1}{5x^2+3x+1}\frac{\mathrm{d}}{\mathrm{d}x}[5x^2+3x+1]\\ &= \frac{10x+3}{5x^2+3x+1}.\end{aligned}$$

   (b) (6 points) $\frac{\mathrm{d}}{\mathrm{d}x}(e^{5x}+e^{7x}+1)^3$

   Solution: Compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(e^{5x}+e^{7x}+1)^3 &= 3(e^{5x}+e^{7x}+1)^2\frac{\mathrm{d}}{\mathrm{d}x}[e^{5x}+e^{7x}+1]\\ &= 3(e^{5x}+e^{7x}+1)^2[5e^{5x}+7e^{7x}].\end{aligned}$$

   (c) (6 points) $\frac{\mathrm{d}}{\mathrm{d}t}\arcsin(7t^3-9)$

   Solution: Compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\arcsin(7t^3-9) &= \frac{1}{\sqrt{1-(7t^3-9)^2}}\frac{\mathrm{d}}{\mathrm{d}t}[7t^3-9]\\ &= \frac{21t^2}{\sqrt{1-(7t^3-9)^2}}.\end{aligned}$$

   (d) (6 points) $\frac{\mathrm{d}}{\mathrm{d}z}3^{5z-1}$

   Solution: Note that

$$(*) \qquad 3^{5z-1} = e^{\log(3^{5z-1})} = e^{(5z-1)\log(3)},$$

   so compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}3^{5z-1} &= \frac{\mathrm{d}}{\mathrm{d}z}e^{(5z-1)\log(3)}\\ &= e^{(5z-1)\log(3)}\frac{\mathrm{d}}{\mathrm{d}z}[(5z-1)\log(3)]\\ &\overset{(*)}{=} 5\log(3)3^{5z-1}.\end{aligned}$$

   (e) (6 points) $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cosh(t)}{\sinh(t)+1}$

   Solution: Compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cosh(t)}{\sinh(t)+1} &= \frac{(\sinh(t)+1)\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t)-\cosh(t)\frac{\mathrm{d}}{\mathrm{d}t}[\sinh(t)+1]}{(\sinh(t)+1)^2}\\ &= \frac{\sinh(t)(\sinh(t)+1)-\cosh^2(t)}{(\sinh(t)+1)^2}.\end{aligned}$$

   (f) (6 points) $\frac{\mathrm{d}}{\mathrm{d}w}\log(\log(\log(w)))$

   Solution: Calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w}\log(\log(\log(w))) &= \frac{1}{\log(\log(w))}\frac{\mathrm{d}}{\mathrm{d}w}\log(\log(w))\\ &= \frac{1}{\log(\log(w))}\frac{1}{\log(w)}\frac{\mathrm{d}}{\mathrm{d}w}\log(w)\\ &= \frac{1}{\log(\log(w))}\frac{1}{\log(w)}\frac{1}{w}.\end{aligned}$$

2. (36 points) Compute the following indefinite integrals.

(a) (6 points) $\int \frac{1}{2+x^2}\mathrm{d}x$

Solution: Using the formula

$$(**) \qquad \int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C,$$

we compute

$$\begin{aligned}\int \frac{1}{2+x^2}\mathrm{d}x &= \int \frac{1}{(\sqrt{2})^2+x^2}\mathrm{d}x \\ &\overset{(**)}{=} \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.\end{aligned}$$

(b) (6 points) $\int \frac{w-5}{w^2+1}\mathrm{d}w$

Solution: First note that we may split this into two integrals:

$$\int \frac{w-5}{w^2+1}\mathrm{d}w = \int \frac{w}{w^2+1}\mathrm{d}w - 5\int \frac{1}{w^2+1}\mathrm{d}w.$$

For the first integral, make the substitution $u = w^2+1$ to arrive at $\frac{1}{2}\mathrm{d}u = w\mathrm{d}w$ and for the second integral, use $(**)$ to calculate

$$\begin{aligned}\int \frac{w-5}{w^2+1}\mathrm{d}w &= \int \frac{w}{w^2+1}\mathrm{d}w - 5\int \frac{1}{w^2+1}\mathrm{d}w \\ &= \frac{1}{2}\int \frac{1}{u}\mathrm{d}u - 5\arctan(w) + C \\ &= \frac{1}{2}\log(u) - 5\arctan(w) + C \\ &= \frac{1}{2}\log(w^2+1) - 5\arctan(w) + C.\end{aligned}$$

(c) (6 points) $\int \frac{4v+4}{2v^2+4v+18}\mathrm{d}v$

Solution: Let $u = 2v^2+4v+18$ so that $\mathrm{d}u = 4v+4\mathrm{d}v$. Now calculate

$$\begin{aligned}\int \frac{4v+4}{2v^2+4v+18}\mathrm{d}v &= \int \frac{1}{u}\mathrm{d}u \\ &= \log(u) + C \\ &= \log(2v^2+4v+18) + C.\end{aligned}$$

(d) (6 points) $\int \coth(\theta)\mathrm{d}\theta \left(\text{recall: } \coth = \frac{\cosh}{\sinh}\right)$

Solution: Take note that if $u = \sinh(\theta)$ then $\mathrm{d}u = \cosh(\theta)\mathrm{d}\theta$. Now calculate

$$\begin{aligned}\int \coth(\theta)\mathrm{d}\theta &= \int \frac{\cosh(\theta)}{\sinh(\theta)}\mathrm{d}\theta \\ &= \int \frac{1}{u}\mathrm{d}u \\ &= \log(u) + C \\ &= \log(\sinh(\theta)) + C.\end{aligned}$$

(e) (6 points) $\int \frac{\cosh(t)}{\sinh(t)+1}\mathrm{d}t$

Solution: Take note that if $u = \sinh(t) + 1$ then $\mathrm{d}u = \cosh(t)\mathrm{d}t$. Now compute

$$\begin{aligned}\int \frac{\cosh(t)}{\sinh(t)+1}\mathrm{d}t &= \int \frac{1}{u}\mathrm{d}u \\ &= \log(u) + C \\ &= \log(\sinh(t)+1) + C.\end{aligned}$$

(f) (6 points) $\displaystyle\int \frac{17}{\sqrt{1-2w^2}}\mathrm{d}w$

Solution: Recall the formula

$$(***) \qquad \int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C.$$

Let $u = \sqrt{2}w$ so that $\frac{1}{\sqrt{2}}\mathrm{d}u = \mathrm{d}w$. Then compute

$$\begin{aligned}\int \frac{17}{\sqrt{1-2w^2}}\mathrm{d}w &= 17\int \frac{1}{\sqrt{1-(\sqrt{2}w)^2}}\mathrm{d}w \\ &= \frac{17}{\sqrt{2}}\int \frac{1}{\sqrt{1-u^2}}\mathrm{d}u \\ &\overset{(***)}{=} \frac{17}{\sqrt{2}}\arcsin(u)\mathrm{d}u + C \\ &= \frac{17}{\sqrt{2}}\arcsin\left(\sqrt{2}w\right) + C.\end{aligned}$$

3. (28 points) Compute the following definite integrals.

(a) (7 points) $\displaystyle\int_0^1 7^x - 11^x \mathrm{d}x$

Solution: Note that $7^x = e^{x\log(7)}$ and $11^x = e^{x\log(11)}$. We will use the substitutions $u = x\log(7)$ so $\frac{1}{\log(7)}\mathrm{d}u = \mathrm{d}x$ and $w = x\log(11)$ so that $\frac{1}{\log(11)}\mathrm{d}w = \mathrm{d}x$. Now compute

$$\begin{aligned}\int_0^1 7^x - 11^x\mathrm{d}x &= \int_0^1 7^x\mathrm{d}x - \int_0^1 11^x\mathrm{d}x \\ &= \int_0^1 e^{x\log(7)}\mathrm{d}x - \int_0^1 e^{x\log(11)}\mathrm{d}x \\ &= \frac{1}{\log(7)}\int_0^{\log(7)} e^u\mathrm{d}u - \frac{1}{\log(11)}\int_0^{\log(11)} e^w\mathrm{d}w \\ &= \frac{1}{\log(7)}\left[e^u\Big|_0^{\log(7)}\right] - \frac{1}{\log(11)}\left[e^u\Big|_0^{\log(11)}\right] \\ &= \frac{1}{\log(7)}(e^{\log(7)}-1) - \frac{1}{\log(11)}(e^{\log(11)}-1) \\ &= \frac{6}{\log(7)} - \frac{10}{\log(11)}.\end{aligned}$$

(b) (7 points) $\displaystyle\int_2^3 \frac{1}{\sqrt{16y^2+1}}\mathrm{d}y$

Solution: Recall the formula

$$(****) \qquad \int \frac{1}{\sqrt{a^2+x^2}}\mathrm{d}x = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C.$$

We shall use the substitution $u = 4y$, hence $\frac{1}{4}\mathrm{d}u = \mathrm{d}y$. Now compute

$$\begin{aligned}
\int_2^3 \frac{1}{\sqrt{16y^2+1}}\mathrm{d}y &= \int_2^3 \frac{1}{\sqrt{(4y)^2+1}}\mathrm{d}y \\
&= \frac{1}{4}\int_8^1 2\frac{1}{\sqrt{u^2+1}}\mathrm{d}u \\
&\overset{(****)}{=} \operatorname{arcsinh}(u)\Big|_2^3 \\
&= \operatorname{arcsinh}(3) - \operatorname{arcsinh}(2).
\end{aligned}$$

(c) (7 points) $\int_0^3 \xi^4 + 3\xi^2 + \xi + 1\mathrm{d}\xi$

Solution: This is just the integral of a polynomial. So compute

$$\begin{aligned}
\int_0^3 \xi^4 + 3\xi^2 + \xi + 1\mathrm{d}\xi &= \frac{\xi^5}{5} + \xi^3 + \frac{\xi^2}{2} + \xi\Big|_0^3 \\
&= \frac{3^5}{5} + 3^3 + \frac{3^2}{2} + 3.
\end{aligned}$$

(d) (7 points) $\int_1^2 \frac{1}{57-4z^2}\mathrm{d}z$

Solution: Recall that

$$(*****) \int \frac{1}{a^2-x^2}\mathrm{d}x = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C.$$

We will use the substitution $u = 2z$ so that $\frac{1}{2}\mathrm{d}u = \mathrm{d}z$. Now compute

$$\begin{aligned}
\int_1^2 \frac{1}{57-4z^2}\mathrm{d}z &= \int_1^2 \frac{1}{(\sqrt{57})^2-(2z)^2}\mathrm{d}z \\
&= \frac{1}{2}\int_2^4 \frac{1}{(\sqrt{57})^2-u^2}\mathrm{d}u \\
&\overset{(*****)}{=} \frac{1}{2\sqrt{57}}\operatorname{arctanh}\left(\frac{u}{\sqrt{57}}\right)\Bigg|_2^4 \\
&= \frac{\operatorname{arctanh}\left(\frac{4}{\sqrt{57}}\right) - \operatorname{arctanh}\left(\frac{2}{\sqrt{57}}\right)}{2\sqrt{57}}.
\end{aligned}$$