Homework 6 - MATH 2200 Spring 2017

1. Write the sentence using quantifiers and predicates. You are welcome to specify a universe in each part to simplify your formulas. Clearly specify the predicates you use.

(a) "All real numbers are complex numbers."
Solution: We read the sentence semi-formally as

"for all real numbers x, x is a complex number"

Let $\mathbb{R}$ be the predicate "is a real number" and let $\mathbb{C}$ be the predicate "is a complex number". We write

$$(\forall x)(\mathbb{R}x \to \mathbb{C}x).$$

(b) "The sum of the angles in any triangle is 180°."
Solution: We read this sentence semi-formally as

"For all triangles with and for all angles a, b, and c of the triangle, where a, b, and c are all different, the sum of the angles a, b, and c is 180°."

Let T be the predicate "is a triangle", let $Etabc$ be the predicate "a, b, and c are the three different angles of the triangle t", and let $Sabc$ be the predicate "the sum of the angles a, b, and c is 180°". Then we may write

$$(\forall t)(\forall a)(\forall b)(\forall c)((Tt \wedge Etabc) \to Sabc).$$

(c) "All Lipschitz functions with compact support are absolutely continuous and of bounded variation."
Solution: We read the sentence semi-formally as

"For all f, if f is a Lipschitz function with compact support, then f is absolutely continuous and f is of bounded variation."

Let L be the predicate "is a Lipschitz function with compact support", let A be the predicate "is absolutely continuous", and let B be the predicate "is of bounded support". We write

$$(\forall f)(Lf \to (Af \wedge Bf)).$$

(d) "There is a continuous function that is uniformly-continuous but is not Hölder-continuous."
Solution: We read this semi-formally as

"There exists an f such that f is continuous, and f is both uniformly-continuous and not Hölder-continuous."

Let C be the predicate "is a continuous function", let U be the predicate "is uniformly-continuous" and, let H be the predicate "is Hölder continuous". We write

$$(\exists f)(Cf \wedge (Uf \wedge \neg Hf)).$$

(e) "Every even integer greater than 2 can be expressed as the sum of two prime numbers." (this is the famous "Goldbach's conjecture"...unknown if it is true or not!)
Solution: We read this semi-formally as

"For all n, if n is both an integer and greater than 2, then n can be expressed as the sum of two prime numbers."

Let I be the predicate "is an integer", G be the predicate "greater than 2", and S be the predicate "can be expressed as the sum of two prime numbers". We write

$$(\forall n)((In \wedge Gn) \to Sn).$$

2. Let R be the predicate "is a real number" and let I be the predicate "is an integer". Is the sentence true or false? If false, explain why.

(a) $(\forall x)(Rx \to x^2 \geq 0)$
Solution: **It is true.** If x is a real number, then $x^2 > 0$ because one of three cases are true: (i) x is positive (hence $x^2 \geq 0$ because the product of two positive numbers is positive), (ii) x is negative (hence $x^2 > 0$ because the product of two negative numbers is positive), or (iii) $x = 0$ (hence $x^2 = 0$ because $0 \cdot 0 = 0$).

(b) $(\exists x)(Ix \wedge x = \sqrt{2})$
Solution: **This is false.** The number $\sqrt{2}$ is not an integer.

(c) $(\forall x)(x^2 + y^2 = 1 \to x^2 = 1 - y^2)$
Solution: **This is true.** If $x^2 + y^2 = 1$, the "laws of algebra" tell us that we may subtract y^2 from both sides and simplify to achieve the formula $x^2 = 1 - y^2$.

(d) $(\forall x)((x > 3) \to (\exists y)(Iy \wedge x + y = \frac{1}{2}))$
Solution: **This is false.** Consider a counterexample by letting $x = 3 + \dfrac{1}{3}$. This x obeys the predicate $x > 3$. For the sentence to be true, we must argue that there is an *integer* y that solves the equation $3 + \dfrac{1}{3} + y = \dfrac{1}{2}$. But the laws of algebra allow us to subtract $3 + \dfrac{1}{3}$ from both sides and simplify to arrive at the equation $y = \dfrac{1}{2} - 3 - \dfrac{1}{3} = \dfrac{1}{6}$. We observe that $\dfrac{1}{6}$ is not an integer. Moreover, we accept (via the theory of algebra) that this value for y is the "only possible value y may take" and so we are certain that an integer found by some other method also will not work.
(note: More specifically, first we acknowledge that the integers "are" or at least "are encoded as" complex numbers – this is similar to recognizing that in a programming language, a variable defined by the formula "`int x = 0`" is "the same, yet different" than "`float x = 0`". We "naturally" frame the equation $3 + \dfrac{1}{3} + y = \dfrac{1}{2}$ as an equation where y is a variable that ranges over the complex numbers and we have the fundamental theorem of algebra to guarantee "uniqueness" of the solution in the sense that this is a 1st degree equation and so the FTA tells us that there is exactly one solution (in the complex numbers, which "includes" the integers). The FTA does not exist in number universes "weaker" than the complex numbers!)

3. Consider the possible universes of quantification: the natural numbers $\mathbb{N}$ (i.e. the numbers $0, 1, 2, \ldots$), the integers $\mathbb{Z}$ (i.e. the numbers $\ldots -1, 0, 1, 2, \ldots$), the real numbers $\mathbb{R}$ (i.e. everything on the number line), and complex numbers $\mathbb{C}$ (i.e. the numbers $a + bi$ where $i = \sqrt{-1}$ and a and b are real numbers).
List all universes (if any) in which the given formula is true.

(a) Let Pxy denote "is a solution of $x + 2 = y$": $(\forall x)(\exists y)(Pxy)$
Solution: This is true in all of the universes $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$.

(b) Let Qxy denote "is a solution of $2x + 3y = 1$": $(\forall y)(\exists x)(Qxy)$
Solution: This is true in only the universes $\mathbb{R}$ and $\mathbb{C}$. To see why it is not true in both $\mathbb{N}$ and $\mathbb{Z}$, consider when the quantifier $(\forall x)$ chooses the value $x = 0$ (0 is in both $\mathbb{N}$ and $\mathbb{Z}$). Then the sentence is claiming $(\exists y)(2(0) + 3y = 1)$. But the only possible y that could satisfy the equation is "$y = \dfrac{1}{3}$" which is not in $\mathbb{N}$ and not in $\mathbb{Z}$.

(c) Let Rx denote "is a solution of $x^2 + 2 = 0$": $(\exists x)(Rx)$
Solution: The only universe this holds in is $\mathbb{C}$. Basic algebra tells us that the solution "should be" given by subtracting 2 to get $x^2 = -2$ and taking square roots giving two solutions of the form $x = \pm\sqrt{-2} = \pm\sqrt{2}i$, where $i = \sqrt{-1}$ is the "imaginary number". Since these are the only "possible" values for x to be and these values are not in $\mathbb{N}, \mathbb{Z}$, or $\mathbb{R}$, we must exclude them from the list.

(d) Let Exy denote "x is equal to y" and $Bzxy$ denote "z is a number between x and y":
$(\forall x)(\forall y)(\neg Exy \to (\exists z)(Bzxy))$
Solution: This holds for $\mathbb{R}$ and $\mathbb{C}$. It does not hold for either $\mathbb{N}$ or $\mathbb{Z}$ because, for instance, if the quantifier $(\forall x)$ chooses $x = 0$ and the quantifier $(\forall y)$ chooses $y = 1$, then these choices satisfy the predicate $\neg Exy$. However there does not exist a member of $\mathbb{N}$ or a member of $\mathbb{Z}$ between 0 and 1.

4. Diagram the scope of every quantifier that appears in the symbolized form of the sentence. Explain whether each instance of a variable is bound or unbound (i.e. free).

(a) "for all x, there exists a y such that $y < z$ and $x < y$" can be symbolized (with appropriate predicate symbols) as:

$$(\forall x)(\exists y)(Lyz \land Lxy)$$

Solution:

$$(\forall x)\,(\exists y)(Lyz \land Lxy)$$

All variables except for the z appearing in the predicate Lyz are bound. The z is free.

(b) The definition of compactness in topology states
"For every open cover u there is a finite subcover f of u,"
and can be symbolized (with appropriate predicate symbols) as:

$$(\forall u)(Ou \to (\exists f)(Sfu))$$

Solution:

$$(\forall u)(Ou \to (\exists f)(Sfu))$$

All variables are bound.

(c) Hölder's inequality in L^p spaces states
"For all measurable functions f and g, for all $p \geq 1$ and for all $q \geq 1$, if $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$,"
and can be symbolized (with appropriate predicate symbols) as:

$$(\forall f)(\forall g)\,[(Mf \land Mg) \to [(\forall p)(\forall q)[(Gp \land Gq) \to (Spq \to Lfg)]]]$$

Solution:

$$(\forall f)\,(\forall g)\left[(Mf \land Mg) \to [(\forall p)\,(\forall q)[(Gp \land Gq) \to (Spq \to Lfg)]]\right]$$

All variables are bound.