

MATH 2200 - EXAM 2 SPRING 2017

SOLUTION

Thursday 9 March 2017

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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (6 points) Symbolize the following sentences. Clearly specify any sentences or predicates used (if needed).

(a) (2 points) “All pears are fruits.”

Solution: Let P be the predicate “is a pear” and let F be the predicate “is a fruit”. Then we can write $(\forall x)(Px \rightarrow Fx)$

(b) (2 points) “If Franz Ferdinand was not shot, then Austria-Hungary would not have mobilized.”

Solution: Let F be “Franz Ferdinand was not shot” and let A be “Austria-Hungary would not have mobilized”. Then this sentence can be symbolized as $F \rightarrow A$ (note: no quantifiers were needed here...).

(c) (2 points) “All transcendental numbers are irrational numbers and $\sqrt{2}$ is an irrational number that is not transcendental.”

Solution: Let T be the predicate “is a transcendental number” and let I be the predicate “is an irrational number”. Let s denote the number $\sqrt{2}$. Then the sentence may be symbolized by

$$(\forall x)(Tx \rightarrow Ix) \wedge (Is \wedge \neg Ts).$$

2. (9 points) True or false? Let R be the predicate “is a real number” and let I be the predicate “is an integer”.

(a) (3 points) $(\forall x)(Rx \rightarrow Ix)$

Solution: False, there are real numbers that are not an integer: for example, $\sqrt{2}$.

(b) (3 points) $(\forall x)(Rx \rightarrow (\exists y)(Iy \wedge (x + y = \sqrt{2})))$

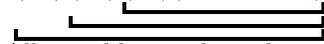
Solution: False, consider, for example, $x = \sqrt{3}$. Then there is no integer y so that $\sqrt{3} + y = \sqrt{2}$ (y would have to be $\sqrt{2} - \sqrt{3}$ – not an integer!).

(c) (3 points) $(\exists x)(Rx \wedge (\forall y)(Iy \rightarrow (x < y^2)))$

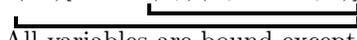
Solution: This is true, for example, take $x = -1$. Then for any integer y , we know that y^2 is zero or positive. So, $x = -1 < 0 \leq y^2$, or in other words, $x < y^2$.

3. (6 points) Diagram the scope of all quantifiers that appear and label every instance of a variable as free or bound.

(a) (3 points) $(\forall x)(\forall y)(\exists z)(Lzx \leftrightarrow Lzy) \rightarrow Exy$

Solution: 
All variables are bound except for the x and y in the predicate E .

(b) (3 points) $(\exists x)[Lzx \wedge (\forall y)(Lyx \rightarrow Sxy)]$

Solution: 
All variables are bound except for

4. (9 points) Formula or not? If so, state so. If not, circle the problem(s). Find all errors for full credit.

(a) (3 points) $(\forall x)(Qz \rightarrow (\exists z)(Pz \wedge Pq)) \leftrightarrow (\exists q)(Px)$

Solution: It is a formula.

(b) (3 points) $Px \leftrightarrow (\forall (\wedge))(\exists (\rightarrow))(Qw)$

Solution: Not a formula.

(c) (3 points) $(\forall x)((\circlearrowleft) \leftrightarrow (\exists z)(Yz \wedge Bz)) \leftrightarrow (Pz \wedge Uz (\wedge) \rightarrow (\exists p)(Tp))$

Solution: Not a formula.

5. (12 points) If the argument is invalid, show it by choosing truth values for all sentences that make all premises true and the conclusion false. If it is valid, write a formal deduction for it.

(a) (6 points) An argument claims to deduce $\neg A \vee \neg N$ from the following premises:

{1}	(1)	$N \rightarrow \neg J$	Premise
{2}	(2)	$J \rightarrow D$	Premise
{3}	(3)	$D \rightarrow \neg A$	Premise

Solution: The argument is invalid. To see it, let A be true, N be true, J be false, and D be false. Then $N \rightarrow \neg J$ is true, $J \rightarrow D$ is true, and $D \rightarrow \neg A$ is true while $\neg A \vee \neg N$ is false.

(b) (6 points) An argument claims to deduce $S \rightarrow B$ from the following premises:

{1}	(1)	$S \rightarrow L$	Premise
{2}	(2)	$\neg L \vee B$	Premise

Solution: It is valid. To see it, write a deduction:

{1}	(1)	$S \rightarrow L$	Premise
{2}	(2)	$\neg L \vee B$	Premise
{2}	(3)	$L \rightarrow B$	2 T Law of Equivalence of Implication and Disjunction
{1, 2}	(4)	$S \rightarrow B$	1 3 T Law of Hypothetical Syllogism

6. (12 points) Are the premises consistent or inconsistent? If inconsistent, derive a contradiction. If consistent, give a true sentential interpretation to prove it.

(a) (6 points) Consider the premises

{1}	(1)	$P \vee Q$	Premise
{2}	(2)	$(P \wedge Q) \rightarrow S$	Premise
{3}	(3)	$\neg S \wedge (\neg Q)$	Premise

Solution: These premises are consistent. To see it, let S be false, Q be false, and P be true. Then $P \vee Q$ is true, $(P \wedge Q) \rightarrow S$ is true (since $P \wedge Q$ is false), and $\neg S \wedge (\neg Q)$ is true (because both $\neg S$ and $\neg Q$ are true).

(b) (6 points) Consider the premises

{1}	(1)	$P \wedge Q$	Premise
{2}	(2)	$(P \wedge Q) \rightarrow (W \wedge T)$	Premise
{3}	(3)	$W \rightarrow \neg(P \wedge Q)$	Premise

Solution: These premises are not consistent. To see it we shall derive a contradiction:

{1}	(1)	$P \wedge Q$	Premise
{2}	(2)	$(P \wedge Q) \rightarrow (W \wedge T)$	Premise
{3}	(3)	$W \rightarrow \neg(P \wedge Q)$	Premise
{1, 2}	(4)	$W \wedge T$	1 2 T Law of Detachment
{1, 2}	(5)	W	4 T Law of Simplification
{1, 2, 3}	(6)	$\neg(P \wedge Q)$	3 5 T Law of Detachment
{1, 2, 3}	(7)	$(P \wedge Q) \wedge \neg(P \wedge Q)$	1 6 T Law of Adjunction

7. (12 points) Consider the following universes of numbers: the natural numbers \mathbb{N} (the numbers $\{0, 1, 2, 3, \dots\}$), the integers \mathbb{Z} (the numbers $\{\dots, -1, 0, 1, 2, \dots\}$), the real numbers \mathbb{R} (all those numbers on the number line), and the complex numbers \mathbb{C} (numbers of the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$).

(a) (4 points) $(\forall x)(\exists y)(x = 5y)$

Solution: It does not hold in \mathbb{N} or \mathbb{Z} : consider when $x = 1$, then no y can be found such that $5y = 1$.

It does hold in \mathbb{R} and \mathbb{C} : we can always pick $y = \frac{x}{5}$ in those universes.

(b) (4 points) $(\forall x)(\forall y)(x + y = y + x)$

Solution: It is true in all universes \mathbb{N} , \mathbb{Z} , \mathbb{R} , and \mathbb{C} .

(c) (4 points) $(\exists x)[\neg(\exists y)(x + y = 0)]$

Solution: It is not true in \mathbb{Z} , \mathbb{R} , or \mathbb{C} : given any x , we can always pick $y = -x$. It is true for \mathbb{N} , for example let $x = 1$ then there is no natural number y such that $x + y = 0$.

8. (15 points) Conclude $\neg Pw$ from the following premises:

{1}	(1) $(\forall x)(Px \rightarrow \neg Sx)$	Premise
{2}	(2) $(\forall x)(\neg Sx \rightarrow Qx)$	Premise
{3}	(3) $\neg Qw$	w Premise

Solution: Calculate

{1}	(1) $(\forall x)(Px \rightarrow \neg Sx)$	Premise
{2}	(2) $(\forall x)(\neg Sx \rightarrow Qx)$	Premise
{3}	(3) $\neg Qw$	w Premise
{1}	(4) $Pw \rightarrow \neg Sw$	1 US
{2}	(5) $\neg Sw \rightarrow Qw$	2 US
{2}	(6) $\neg Qw \rightarrow \neg(\neg Sw)$	5 Law of Contraposition
{1}	(7) $\neg(\neg Sw) \rightarrow \neg Pw$	4 Law of Contraposition
{2, 3}	(8) $\neg(\neg Sw)$	w 3 6 Law of Detachment
{1, 2, 3}	(9) $\neg Pw$	w 7 8 Law of Detachment

9. (19 points) Use the following premises **and reductio ad absurdum** to deduce $\neg P$:

{1}	(1) $S \rightarrow \neg P$	Premise
{2}	(2) $\neg S \rightarrow \neg Q$	Premise
{3}	(3) $Q \vee \neg P$	Premise

Solution:

{1}	(1) $S \rightarrow \neg P$	Premise
{2}	(2) $\neg S \rightarrow \neg Q$	Premise
{3}	(3) $Q \vee \neg P$	Premise
{4}	(4) $\neg(\neg P)$	Premise (for contradiction)
{1}	(5) $\neg(\neg P) \rightarrow \neg S$	1 T Law of Contraposition
{1, 4}	(6) $\neg S$	4 5 T Law of Detachment
{1, 2, 4}	(7) $\neg Q$	2 6 T Law of Detachment
{1, 2, 3, 4}	(8) $\neg P$	3 7 T Modus tollendo ponens
{1, 2, 3, 4}	(9) $\neg P \wedge \neg(\neg P)$	4 8 T Law of Adjunction
{1, 2, 3}	(10) $\neg P$	4 9 T RAA

TABLE OF USEFUL TAUTOLOGIES

TAUTOLOGICAL IMPLICATIONS

<p>Law of Detachment <i>Modus tollendo tollens</i> <i>Modus tollendo ponens</i> Law of Simplification Law of Adjunction Law of Hypothetical Syllogism Law of Exportation Law of Importation Law of Absurdity Law of Addition</p>	<p>$P \& (P \rightarrow Q) \rightarrow Q$ $\neg Q \& (P \rightarrow Q) \rightarrow \neg P$ $\neg P \& (P \vee Q) \rightarrow Q$ $P \& Q \rightarrow P$ $P \& Q \rightarrow P \& Q$ $(P \rightarrow Q) \& (Q \rightarrow R) \rightarrow (P \rightarrow R)$ $[P \& Q \rightarrow R] \rightarrow [P \rightarrow (Q \rightarrow R)]$ $[P \rightarrow (Q \rightarrow R)] \rightarrow [P \& Q \rightarrow R]$ $[P \rightarrow Q \& \neg Q] \rightarrow \neg P$ $P \rightarrow P \vee Q$</p>
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TAUTOLOGICAL EQUIVALENCES

<p>Law of Double Negation Law of Contraposition De Morgan's Laws Commutative Laws Law of Equivalence for Implication and Disjunction Law of Negation for Implication A Law for Biconditional Sentences Another Law for Biconditional Sentences</p>	<p>$P \leftrightarrow \neg\neg P$ $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$ $\neg(P \& Q) \leftrightarrow \neg P \vee \neg Q$ $\neg(P \vee Q) \leftrightarrow \neg P \& \neg Q$ $P \& Q \leftrightarrow Q \& P$ $P \vee Q \leftrightarrow Q \vee P$ $(P \rightarrow Q) \leftrightarrow \neg P \vee Q$ $\neg(P \rightarrow Q) \leftrightarrow P \& \neg Q$ $(P \leftrightarrow Q) \leftrightarrow (P \rightarrow Q) \& (Q \rightarrow P)$ $(P \leftrightarrow Q) \leftrightarrow (P \& Q) \vee (\neg P \& \neg Q)$</p>
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TWO FURTHER TAUTOLOGIES

<p>Law of Excluded Middle Law of Contradiction</p>	<p>$P \vee \neg P$ $\neg(P \& \neg P)$</p>
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