MATH 2200 - EXAM 2 SPRING 2017
SOLUTION

Thursday 9 March 2017
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (6 points) Symbolize the following sentences. Clearly specify any sentences or predicates used (if needed).

   (a) (2 points) "All pears are fruits."
   Solution: Let P the predicate "is a pear" and let F be the predicate "is a fruit". Then we can write $(\forall x)(Px \to Fx)$

   (b) (2 points) "If Franz Ferdinand was not shot, then Austria-Hungary would not have mobilized."
   Solution: Let F be "Franz Ferdinand was not shot" and let A be "Austria-Hungary would not have mobilized". Then this sentence can be symbolized as $F \to A$ (note: no quantifiers were needed here...).

   (c) (2 points) "All transcendental numbers are irrational numbers and $\sqrt{2}$ is an irrational number that is not transcendental."
   Solution: Let T be the predicate "is a transcendental number" and let I be the predicate "is an irrational number". Let s denote the number $\sqrt{2}$. Then the sentence may be symbolized by

$$(\forall x)(Tx \to Ix) \wedge (Is \wedge \neg Ts).$$

2. (9 points) True or false? Let R be the predicate "is a real number" and let I be the predicate "is an integer".

   (a) (3 points) $(\forall x)(Rx \to Ix)$
   Solution: False, there are real numbers that are not an integer: for example, $\sqrt{2}$.

   (b) (3 points) $(\forall x)\left(Rx \to (\exists y)(Iy \wedge \left(x + y = \sqrt{2}\right))\right)$
   Solution: False, consider, for example, $x = \sqrt{3}$. Then there is no integer y so that $\sqrt{3} + y = \sqrt{2}$ (y would have to be $\sqrt{2} - \sqrt{3}$ – not an integer!).

   (c) (3 points) $(\exists x)(Rx \wedge (\forall y)(Iy \to (x < y^2)))$
   Solution: This is true, for example, take $x = -1$. Then for any integer y, we know that y^2 is zero or positive. So, $x = -1 < 0 \leq y^2$, or in other words, $x < y^2$.

3. (6 points) Diagram the scope of all quantifiers that appear and label every instance of a variable as free or bound.

   (a) (3 points) $(\forall x)(\forall y)(\exists z)(Lzx \leftrightarrow Lzy) \to Exy$
   Solution: All variables are bound except for the x and y in the predicate E.

   (b) (3 points) $(\exists x)[Lzx \wedge (\forall y)(Lyx \to Syx)]$
   Solution: All variables are bound except for

4. (9 points) Formula or not? If so, state so. If not, circle the problem(s). Find all errors for full credit.

   (a) (3 points) $(\forall x)(Qz \to (\exists z)(Pz \wedge Pq)) \leftrightarrow (\exists q)(Px)$
   Solution: It is a formula.

   (b) (3 points) $Px \leftrightarrow (\forall \wedge)(\exists \to)(Qw)$
   Solution: Not a formula.

   (c) (3 points) $(\forall x)((x \leftrightarrow (\exists z)(Yz \wedge Bz)) \leftrightarrow (Pz \wedge Uz \wedge \to (\exists p)(Tp))$
   Solution: Not a formula.

5. (12 points) If the argument is invalid, show it by choosing truth values for all sentences that make all premises true and the conclusion false. If it is valid, write a formal deduction for it.

   (a) (6 points) An argument claims to deduce $\neg A \vee \neg N$ from the following premises:

$\{1\}$	(1) $N \to \neg J$	Premise
$\{2\}$	(2) $J \to D$	Premise
$\{3\}$	(3) $D \to \neg A$	Premise

   Solution: The argument is invalid. To see it, let A be true, N be true, J be false, and D be false. Then $N \to \neg J$ is true, $J \to D$ is true, and $D \to \neg A$ is true while $\neg A \vee \neg N$ is false.

   (b) (6 points) An argument claims to deduce $S \to B$ from the following premises:

$\{1\}$	(1) $S \to L$	Premise
$\{2\}$	(2) $\neg L \vee B$	Premise

   Solution: It is valid. To see it, write a deduction:

$\{1\}$	(1) $S \to L$	Premise
$\{2\}$	(2) $\neg L \vee B$	Premise
$\{2\}$	(3) $L \to B$	2 T Law of Equivalence of Implication and Disjunction
$\{1, 2\}$	(4) $S \to B$	1 3 T Law of Hypothetical Syllogism

6. (12 points) Are the premises consistent or inconsistent? If inconsistent, derive a contradiction. If consistent, give a true sentential interpretation to prove it.

   (a) (6 points) Consider the premises

$\{1\}$	(1) $P \vee Q$	Premise
$\{2\}$	(2) $(P \wedge Q) \to S$	Premise
$\{3\}$	(3) $\neg S \wedge (\neg Q)$	Premise

   Solution: These premises are consistent. To see it, let S be false, Q be false, and P be true. Then $P \vee Q$ is true, $(P \wedge Q) \to S$ is true (since $P \wedge Q$ is false), and $\neg S \wedge (\neg Q)$ is true (because both $\neg S$ and $\neg Q$ are true).

   (b) (6 points) Consider the premises

$\{1\}$	(1) $P \wedge Q$	Premise
$\{2\}$	(2) $(P \wedge Q) \to (W \wedge T)$	Premise
$\{3\}$	(3) $W \to \neg(P \wedge Q)$	Premise

   Solution: These premises are not consistent. To see it we shall derive a contradiction:

$\{1\}$	(1) $P \wedge Q$	Premise
$\{2\}$	(2) $(P \wedge Q) \to (W \wedge T)$	Premise
$\{3\}$	(3) $W \to \neg(P \wedge Q)$	Premise
$\{1, 2\}$	(4) $W \wedge T$	1 2 T Law of Detachment
$\{1, 2\}$	(5) W	4 T Law of Simplification
$\{1, 2, 3\}$	(6) $\neg(P \wedge Q)$	3 5 T Law of Detachment
$\{1, 2, 3\}$	(7) $(P \wedge Q) \wedge \neg(P \wedge Q)$	1 6 T Law of Adjunction

7. (12 points) Consider the following universes of numbers: the natural numbers $\mathbb{N}$ (the numbers $\{0, 1, 2, 3, \ldots\}$), the integers $\mathbb{Z}$ (the numbers $\{\ldots, -1, 0, 1, 2, \ldots\}$), the real numbers $\mathbb{R}$ (all those numbers on the number line), and the complex numbers $\mathbb{C}$ (numbers of the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$).

   (a) (4 points) $(\forall x)(\exists y)(x = 5y)$
   Solution: It does not hold in $\mathbb{N}$ or $\mathbb{Z}$: consider when $x = 1$, then no y can be found such that $5y = 1$. It does hold in $\mathbb{R}$ and $\mathbb{C}$: we can always pick $y = \frac{x}{5}$ in those universes.

   (b) (4 points) $(\forall x)(\forall y)(x + y = y + x)$
   Solution: It is true in all universes $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$.

   (c) (4 points) $(\exists x)\,[\neg(\exists y)(x + y = 0)]$
   Solution: It is not true in $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$: given any x, we can always pick $y = -x$. It is true for $\mathbb{N}$, for example let $x = 1$ then there is no natural number y such that $x + y = 0$.

8. (15 points) Conclude $\neg Pw$ from the following premises:

$\{1\}$	(1) $(\forall x)(Px \to \neg Sx)$	Premise
$\{2\}$	(2) $(\forall x)(\neg Sx \to Qx)$	Premise
$\{3\}$	(3) $\neg Qw$	w Premise

Solution: Calculate

$\{1\}$	(1) $(\forall x)(Px \to \neg Sx)$	Premise
$\{2\}$	(2) $(\forall x)(\neg Sx \to Qx)$	Premise
$\{3\}$	(3) $\neg Qw$	w Premise
$\{1\}$	(4) $Pw \to \neg Sw$	1 US
$\{2\}$	(5) $\neg Sw \to Qw$	2 US
$\{2\}$	(6) $\neg Qw \to \neg(\neg Sw)$	5 Law of Contraposition
$\{1\}$	(7) $\neg(\neg Sw) \to \neg Pw$	4 Law of Contraposition
$\{2, 3\}$	(8) $\neg(\neg Sw)$	w 3 6 Law of Detachment
$\{1, 2, 3\}$	(9) $\neg Pw$	w 7 8 Law of Detachment

9. (19 points) Use the following premises **and reductio ad absurdum** to deduce $\neg P$:

$\{1\}$	(1) $S \to \neg P$	Premise
$\{2\}$	(2) $\neg S \to \neg Q$	Premise
$\{3\}$	(3) $Q \vee \neg P$	Premise

Solution:

$\{1\}$	(1) $S \to \neg P$	Premise
$\{2\}$	(2) $\neg S \to \neg Q$	Premise
$\{3\}$	(3) $Q \vee \neg P$	Premise
$\{4\}$	(4) $\neg(\neg P)$	Premise (for contradiction)
$\{1\}$	(5) $\neg(\neg P) \to \neg S$	1 T Law of Contraposition
$\{1, 4\}$	(6) $\neg S$	4 5 T Law of Detachment
$\{1, 2, 4\}$	(7) $\neg Q$	2 6 T Law of Detachment
$\{1, 2, 3, 4\}$	(8) $\neg P$	3 7 T Modus tollendo ponens
$\{1, 2, 3, 4\}$	(9) $\neg P \wedge \neg(\neg P)$	4 8 T Law of Adjunction
$\{1, 2, 3\}$	(10) $\neg P$	4 9 T RAA

TABLE OF USEFUL TAUTOLOGIES

TAUTOLOGICAL IMPLICATIONS

Law of Detachment	$P \mathbin{\&} (P \rightarrow Q) \rightarrow Q$
Modus tollendo tollens	$-Q \mathbin{\&} (P \rightarrow Q) \rightarrow -P$
Modus tollendo ponens	$-P \mathbin{\&} (P \vee Q) \rightarrow Q$
Law of Simplification	$P \mathbin{\&} Q \rightarrow P$
Law of Adjunction	$P \mathbin{\&} Q \rightarrow P \mathbin{\&} Q$
Law of Hypothetical Syllogism	$(P \rightarrow Q) \mathbin{\&} (Q \rightarrow R) \rightarrow (P \rightarrow R)$
Law of Exportation	$[P \mathbin{\&} Q \rightarrow R] \rightarrow [P \rightarrow (Q \rightarrow R)]$
Law of Importation	$[P \rightarrow (Q \rightarrow R)] \rightarrow [P \mathbin{\&} Q \rightarrow R]$
Law of Absurdity	$[P \rightarrow Q \mathbin{\&} -Q] \rightarrow -P$
Law of Addition	$P \rightarrow P \vee Q$

TAUTOLOGICAL EQUIVALENCES

Law of Double Negation	$P \leftrightarrow --P$
Law of Contraposition	$(P \rightarrow Q) \leftrightarrow (-Q \rightarrow -P)$
De Morgan's Laws	$-(P \mathbin{\&} Q) \leftrightarrow -P \vee -Q$
	$-(P \vee Q) \leftrightarrow -P \mathbin{\&} -Q$
Commutative Laws	$P \mathbin{\&} Q \leftrightarrow Q \mathbin{\&} P$
	$P \vee Q \leftrightarrow Q \vee P$
Law of Equivalence for Implication and Disjunction	$(P \rightarrow Q) \leftrightarrow -P \vee Q$
Law of Negation for Implication	$-(P \rightarrow Q) \leftrightarrow P \mathbin{\&} -Q$
A Law for Biconditional Sentences	$(P \leftrightarrow Q) \leftrightarrow (P \rightarrow Q) \mathbin{\&} (Q \rightarrow P)$
Another Law for Biconditional Sentences	$(P \leftrightarrow Q) \leftrightarrow (P \mathbin{\&} Q) \vee (-P \mathbin{\&} -Q)$

TWO FURTHER TAUTOLOGIES

Law of Excluded Middle	$P \vee -P$
Law of Contradiction	$-(P \mathbin{\&} -P)$