MATH 2222HJ Spring 2016
Quiz 5 SOLUTION

Show all work clearly and in order, and circle your final answers.
Justify your answers algebraically whenever possible. Unjustified work may not receive full credit.

1. (*2 points*) Suppose that $f(x,y,z) = 3yze^{2x}$ and $x = 2t$, $y = 4t$, and $z = \sin(2t)$. Compute $\frac{\mathrm{d}f}{\mathrm{d}t}$ whenever $t = \frac{\pi}{2}$.
Solution: By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= 2\left(6yze^{2x}\right) + 4\left(3ze^{2x}\right) + \left(2\cos(2t)\right)\left(3ye^{2x}\right)\end{aligned}$$

When $t = \frac{\pi}{2}$ we have $x = \pi, y = 2\pi$, and $z = 0$, so

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=\frac{\pi}{2}} = 0 + 0 + 6\cos(\pi)(2\pi)e^{2\pi} = -12\pi e^{2\pi}.$$

2. (*3 points*) Consider the function $f(x,y) = 6x^2 + 4xy - 3y^2$ and the point $P = (6,-1)$. Find the direction of steepest ascent at P and find a vector that points in the direction of no change at P.
Solution: First compute

$$\nabla f(x,y) = \langle 12x + 4y, 4x - 6y\rangle .$$

A well-known theorem says that the direction of steepest ascent of f at a point is in the direction of the gradient of f at that point. Hence the direction of steepest ascent at $(6,-1)$ is in the direction of the vector

$$\nabla f(6,-1) = \langle 12(6) + 4(-1), 4(6) - 6(-1)\rangle = \langle 68, 30\rangle .$$

Note: you don't need to normalize this vector unless you plan to compute the "maximum rate of change" in this direction.
It is also well-known that the direction of no change is in the direction orthogonal to the level curve of f that touches the point in question. Recall that level curves are of the form $k = f(x,y)$ for some k. To find k, we plug in the point $(6,-1)$ into this equation to get

$$k = f(6,-1) = 6(6^2) + 4(6)(-1) - 3(-1)^2 = 216 - 24 - 3 = 189,$$

hence the point $(6,-1)$ lies on the level curve given by the equation $189 = f(x,y)$, or equivalently

$$0 = 6x^2 + 4xy - 3y^2 - 189.$$

This is a curve in the plane. We wish to find the slope of the tangent line of this curve at the point $(6,-1)$. To do this, first notice our level curve is of the form $0 = F(x,y)$ where $F(x,y) = 6x^2 + 4xy - 3y^2 - 189$. We use the implicit differentiation formula $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$ to see

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{12x + 4y}{4x - 6y},$$

and so the tangent at $(6,-1)$ has slope

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=6,y=-1} = -\frac{12(6) + 4(-1)}{4(6) - 6(-1)} = -\frac{34}{15}.$$

Recall that the vector $< a, b >$ is parallel to the line with slope $\dfrac{b}{a}$, so we see from this that the vector $< -15, 34 >$ is parallel to the tangent line at $(6, -1)$, i.e. the vector $< -15, 34 >$ is in the direction of no change at $(6, -1)$.

Note: we can check the result for the direction of no change. It means if we take the directional derivative of f in the direction $< -15, 34 >$ at $(6, -1)$, we should get a zero result. Let

$$\vec{u} = \frac{< -15, 34 >}{\| < -15, 34 > \|} = \frac{< -15, 34 >}{\sqrt{(-15)^2 + 34^2}} = \frac{< -15, 34 >}{\sqrt{1381}}.$$

Compute

$$\begin{aligned} D_{\vec{u}} f(x, y) &\stackrel{\text{def}}{=} \nabla f(x, y) \cdot \vec{u} \\ &= \langle 12x + 4y, 4x - 6y \rangle \cdot \frac{< -15, 34 >}{\sqrt{1381}}. \end{aligned}$$

Hence

$$\begin{aligned} D_{\vec{u}} f(6, -1) &= \frac{1}{\sqrt{1381}} \langle 68, 30 \rangle \cdot \langle -15, 34 \rangle \\ &= \frac{1}{\sqrt{1381}} (-1020 + 1020) \\ &= 0, \end{aligned}$$

as was to be shown.