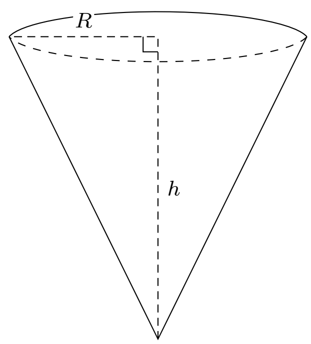
Question from 1PM class on 28 April 2016
Problem: Find the area of a cone (not including the base) with radius r and height h using a parametric description of the surface, where r and h are positive constants.
Solution: First let us draw the cone:



Notice that cross sections of the cone are circles (parallel to the xy-plane), so we want to parametrize circles of some radius dependent somehow on the height. At height $z = 0$ we want a circle with zero radius (i.e. a point) and at height $z = h$ we want height R. Thus it makes sense to write the radius of the circle at height z to be $z\frac{R}{h}$ (notice what happens at height $z = h$). We will use cylindrical coordinates (letting u represent the cylindrical variable θ, letting v represent the cylindrical variable z, and letting the cylindrical variable $r = v\frac{R}{h}$):

$$\begin{cases} \vec{r}(u,v) = \left\langle v\frac{R}{h}\cos(u), v\frac{R}{h}\sin(u), v \right\rangle \\ 0 \le u \le 2\pi, \quad 0 \le v \le h. \end{cases}$$

Therefore we may compute

$$\vec{r}_u = \left\langle -v\frac{R}{h}\sin(u), v\frac{R}{h}\cos(u), 0 \right\rangle,$$

$$\vec{r}_v = \left\langle \frac{R}{h}\cos(u), \frac{R}{h}\sin(u), 1 \right\rangle,$$

$$\vec{r}_u \times \vec{r}_v = \left\langle v\frac{R}{h}\cos(u), v\frac{R}{h}\sin(u), -v\frac{R^2}{h^2} \right\rangle,$$

and hence

$$\begin{aligned} \|\vec{r}_u \times \vec{r}_v\| &= \sqrt{v^2\frac{R^2}{h^2} + v^2\frac{R^4}{h^4}} \\ &= \frac{Rv}{h}\sqrt{1 + \frac{R^2}{h^2}} \\ &= \frac{Rv}{h}\sqrt{\frac{R^2 + h^2}{h^2}} \\ &= \frac{Rv}{h^2}\sqrt{R^2 + h^2}. \end{aligned}$$

Now we may compute

$$\begin{aligned}
\text{SurfaceArea} &= \iint\limits_S 1 dS \\
&= \int_0^{2\pi} \int_0^h \|\vec{r}_u \times \vec{r}_v\| \,\mathrm{d}v\mathrm{d}u \\
&= \frac{R}{h^2}\sqrt{R^2+h^2} \int_0^{2\pi}\int_0^h v\mathrm{d}v\mathrm{d}u \\
&= \frac{R}{h^2}\sqrt{R^2+h^2}\frac{2\pi h^2}{2} \\
&= R\pi\sqrt{R^2+h^2}.
\end{aligned}$$