MATH 1190 - EXAM 4 FALL 2016
SOLUTION

Friday 18 November 2016
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (15 points) Use $n = 3$ rectangles to approximate the area under the function $f(x) = x^2 + 3$ and above the interval $[0, 3]$ **using right endpoints**.

Solution: Using $n = 3$ rectangles requires $\Delta x = \frac{3-0}{3} = 1$. Therefore the right endpoints are at $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. Evaluating f at these points yields

$$f(1) = 1^2 + 3 = 4,$$

$$f(2) = 2^2 + 3 = 7,$$

and

$$f(3) = 3^2 + 3 = 12.$$

Therefore the approximate area is given by

$$\text{Area} \approx \sum_{\mathrm{k}=1}^{3} \mathrm{f}(\mathrm{x_k})\Delta \mathrm{x} = \mathrm{f}(\mathrm{x_1})\Delta\mathrm{x} + \mathrm{f}(\mathrm{x_2})\Delta\mathrm{x} + \mathrm{f}(\mathrm{x_3})\Delta\mathrm{x} = 4(1) + 7(1) + 12(1) = 23.$$

2. (15 points) Compute the following summation using the properties of sums (after applying the sum formula, you do not have to further simplify!):

$$\sum_{k=1}^{50} k^3 + k.$$

Solution: Using the formulas for the sum of cubes and sum of first powers, we get

$$\sum_{k=1}^{50} k^3 + k = \left(\sum_{k=1}^{50} k^3\right) + \left(\sum_{k=1}^{50} k\right) = \frac{50^2(51)^2}{4} + \frac{50(51)}{2}.$$

3. (24 points) Find the following integrals any way you wish:

(a) (6 points) $\int_0^1 x^2 + 3x + 2\mathrm{dx}$

Solution: Calculate

$$\int_0^1 x^2 + 3x + 2\mathrm{dx} = \frac{\mathrm{x}^3}{3} + \frac{3}{2}\mathrm{x}^2 + 2\mathrm{x}\Big|_0^1 = \frac{1}{3} + \frac{3}{2} + 2 = \frac{23}{6}$$

(b) (6 points) $\int_0^{\frac{\pi}{2}} \cos(x)\mathrm{dx}$

Solution: Calculate

$$\int_0^{\frac{\pi}{2}} \cos(x)\mathrm{dx} = \sin(\mathrm{x})\Big|_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$$

(c) (6 points) $\int \frac{x}{\sqrt{x+1}}\mathrm{dx}$

Solution: Let $u = x + 1$ then $\mathrm{du} = \mathrm{dx}$ and $x = u - 1$ so using substitution, we get

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\mathrm{dx} &= \int \frac{u-1}{\sqrt{u}}\mathrm{du} \\ &= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\mathrm{du} \\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C\end{aligned}$$

(d) (6 points) $\int \frac{\sin(x)}{\cos^3(x)}\text{dx}$

Solution: Let $u = \cos(x)$ so du $= -\sin(\text{x})\text{dx}$. Use substitution to get

$$\begin{aligned}\int \frac{\sin(x)}{\cos^3(x)}\text{dx} &= -\int \frac{1}{u^3}\text{du} \\ &= -\int u^{-3}\text{du} \\ &= -\frac{u^{-2}}{-2} + C \\ &= \frac{1}{2\cos^2(x)} + C \\ &= \frac{1}{2}\sec^2(x) + C\end{aligned}$$

4. (20 points) In this problem you will use the **limit definition** of a definite integral to compute

$$\int_0^4 2x + 1\text{dx}.$$

(a) (5 points) What is Δx? What is x_k?

Solution: Here

$$\Delta x = \frac{4-0}{n} = \frac{4}{n}$$

and

$$x_k = 0 + k\Delta x = \frac{4k}{n}.$$

(b) (5 points) What is $f(x_k)$?

Solution: Here $f(x) = 2x + 1$ so using $x_k = \frac{4k}{n}$ we get

$$f(x_k) = 2x_k + 1 = \frac{8k}{n} + 1.$$

(c) (5 points) What is $\sum_{k=1}^{n} f(x_k)\Delta x$?

Solution: Calculate

$$\begin{aligned}\sum_{k=1}^{n} f(x_k)\Delta x &= \frac{4}{n}\sum_{k=1}^{n}\frac{8k}{n} + 1 \\ &= \frac{32}{n^2}\left(\sum_{k=1}^{n} k\right) + \frac{4}{n}\left(\sum_{k=1}^{n} 1\right) \\ &= \frac{32}{n^2}\frac{n(n+1)}{2} + \frac{4}{n}(n) \\ &= 16\left(1 + \frac{1}{n}\right) + 4.\end{aligned}$$

(d) (5 points) Use your answers in parts (a)-(c) to compute $\int_0^4 2x + 1\text{dx}$ using the limit definition.

Solution: The limit definition tells us that

$$\begin{aligned}\int_0^4 2x + 1\text{dx} &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x \\ &= \lim_{n\to\infty} 16\left(1 + \frac{1}{n}\right) + 4 \\ &= 16 + 4 \\ &= 20.\end{aligned}$$

5. (10 points) Find the average value of $f(x) = x^2 + 1$ on the interval $[-2, 3]$.
Solution: The average value is given by

$$\begin{aligned} \text{AvgVal} &= \frac{1}{3-(-2)} \int_{-2}^{3} x^2 + 1\text{dx} \\ &= \frac{1}{5}\left[\frac{x^3}{3} + x\Big|_{-2}^{3}\right. \\ &= \frac{1}{5}\left[\left(\frac{27}{3} + 3\right) - \left(\frac{-8}{3} - 2\right)\right] \end{aligned}$$

6. (15 points) (a) (7 points) Let $F(x) = \int_0^x t^2 \text{dt}$. Compute $F'(x)$.
Solution: Using the fundamental theorem of calculus ("second part") we get

$$F'(x) = x^2.$$

(b) (8 points) Let $F(x) = \int_{\sin(x)}^{x} t^2 \text{dt}$. Compute $F'(x)$.
Solution: First rearrange this integral into two integrals each with the variable x appearing in the top:

$$F(x) = \int_{\sin(x)}^{0} t^2 \text{dt} + \int_0^{\text{x}} \text{t}^2 \text{dt} = -\int_0^{\sin(\text{x})} \text{t}^2\text{dt} + \int_0^{\text{x}} \text{t}^2\text{dt}.$$

Letting $G(x) = \int_0^x t^2 \text{dt}$ we see from the second fundamental theorem of calculus that $G'(x) = x^2$ and also we can write

$$F(x) = -G(\sin(x)) + G(x),$$

so that using the chain rule,

$$\begin{aligned} F'(x) &= -\frac{\text{d}}{\text{dx}} G(\sin(x)) + G'(x) \\ &= -G'(\sin(x))\cos(x) + G'(x) \\ &= -\sin^2(x)\cos(x) + x^2. \end{aligned}$$

Summation formulas (ok to tear this page out):

for any constant α, $\sum_{k=1}^{n} \alpha = n\alpha$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$