MATH 1190 - EXAM 3 FALL 2016
SOLUTION

Friday 28 October 2016
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (15 points)

(a) (5 points) Use the first derivative test to find all relative extrema of the function:

$$f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x.$$

Solution: Compute

$$f'(x) = x^2 + 3x + 2 \overset{\text{set}}{=} 0.$$

Solve this quadratic equation any way you wish (quadratic formula is ok, but it also factors) to get $x = -1, -2$. Pick test points – say $x = -3$, $x = -\frac{3}{2}$ and $x = 0$. Evaluate f at the test points:

$$f(-3) = (-3)^2 + 3(-3) + 2 = 9 - 9 + 2 = 2 > 0,$$

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + 2 = \frac{9}{4} - \frac{9}{2} + 2 < 0,$$

and

$$f(0) = 0^2 + 0 + 2 > 0.$$

Therefore we may conclude that there is a relative minimum at $x = -2$ and a relative maximum at $x = -1$.

(b) (5 points) Use the second derivative test to find all relative extrema of the function

$$f(x) = x^2 + 3x + 2.$$

Solution: Calculate

$$f'(x) = 2x + 3 \overset{\text{set}}{=} 0.$$

This yields the critical point $x = -\frac{3}{2}$. Now compute

$$f''(x) = 2,$$

so

$$f''\left(-\frac{3}{2}\right) = 2 > 0.$$

Therefore $x = -\frac{3}{2}$ is a relative minimum (because f is concave up there!).

(c) (5 points) Find all inflection points of the function

$$f(x) = \frac{4}{x^2+1}.$$

Solution: Compute

$$f'(x) = \frac{0 - 4(2x)}{(x^2+1)^2} = \frac{-8x}{(x^2+1)^2} \overset{\text{set}}{=} 0$$

and

$$\begin{aligned} f''(x) &= \frac{-8(x^2+1)^2 - (-8x)(2(x^2+1)(2x))}{(x^2+1)^4} \\ &= \frac{-8(x^2+1) + 32x^2}{(x^2+1)^3} \\ &= \frac{24x^2 - 8}{(x^2+1)^3} \\ &\overset{\text{set}}{=} 0. \end{aligned}$$

Solve this by first multiplying by $(x^2+1)^3$

$$24x^2 - 8 = 0.$$

Therefore
$$x^2 = \frac{1}{3},$$
and we get
$$x = \pm\frac{1}{\sqrt{3}}.$$
Pick test points $x = -2$, $x = 0$, and $x = 2$. Calculate
$$f''(-2) = \frac{24(4) - 8}{5^3} > 0,$$
$$f''(0) = \frac{-8}{1} < 0,$$
and
$$f''(2) = \frac{24(4) - 8}{5^3} > 0.$$
Therefore f has inflection points at $x = \pm\frac{1}{\sqrt{3}}$.

2. (16 points) Find two positive real numbers with the property that the sum of the square of the first one with the second one is 27 and whose product is a maximum.
Solution: Let x be the first number and y be the second number. The problem tells us that x and y must satisfy the constraint
$$x^2 + y = 27$$
and we are asked to maximize $P = xy$. Solving the constraint for y yields
$$y = 27 - x^2.$$
Plug this into P to get
$$P(x) = x(27 - x^2) = 27x - x^3.$$
Now we want to optimize P. First calculate
$$P'(x) = 27 - 3x^2 \stackrel{\text{set}}{=} 0.$$
This yields
$$9 = x^2,$$
and so
$$x = \pm 3.$$
Since in this problem both x and y are said to be positive, we take the solution $x = 3$. To prove a maximum occurs here, compute
$$P''(x) = -6x,$$
and plug the critical point $x = 3$ into it to get
$$P''(3) = -6(3) = -18 < 0.$$
Therefore by the second derivative test, P has a minimum when $x = 3$. Now find y by going back to the constraint and plugging x into it:
$$y = 27 - 3^2 = 18.$$

3. (16 points) Find the base and height of a right triangle with hypotenuse 1 whose area is 8 and whose perimeter is a minimum.
Solution: Let b denote the base of the triangle and let h denote the height. By the area of a triangle formula, we get the constraint equation
$$\frac{1}{2}bh = 8.$$

We are asked to optimize the perimeter

$$P = b + h + 1.$$

Solve the constraint equation for one of the variables, say b, to get

$$b = \frac{16}{h}.$$

Plug this value into the perimeter equation to get

$$P(h) = \frac{16}{h} + h + 1.$$

Optimize P by first taking the derivative:

$$P'(h) = -\frac{16}{h^2} + 1 \overset{\text{set}}{=} 0.$$

Solve this equation to get

$$16 = h^2,$$

or

$$h = \pm 4.$$

Since h is a geometrical length, we take the positive solution $h = 4$. We must prove that P is a minimum when $h = 4$. To do that first compute

$$P''(h) = \frac{32}{h^3}$$

and plug in the critical point $h = 4$ into it to get

$$P''(4) = \frac{32}{4^3} > 0,$$

from which we use the second derivative test to conclude that P has a relative minimum at $h = 4$. Now find the corresponding value of b by plugging this value of h into the constraint equation:

$$b = \frac{16}{4} = 4.$$

4. (15 points)
   (a) (6 points) Find all functions F with the property that $F'(x) = 1$. In other words, find $D^{-1}(1)$.
   Solution: By the rules of anti-differentiation,

   $$D^{-1}(1) = x + C,$$

   where C denotes an arbitrary constant.
   (b) (9 points) Find all functions F with the property that $F'(x) = 7x^4 + 11x^3 + x + 10$. In other words, find $D^{-1}(7x^4 + 11x^3 + x + 10)$.
   Solution: Using the rules of anti-differentiation,

   $$D^{-1}(7x^4 + 11x^3 + x + 10) = \frac{7}{5}x^5 + \frac{11}{4}x^4 + \frac{1}{2}x^2 + 10x + C,$$

   where C denotes an arbitrary constant.

5. (10 points) Compute the sum.
   (a) (4 points) $\sum_{k=0}^{3} 1$
   Solution: Calculate

   $$\sum_{k=0}^{3} 1 = 1 + 1 + 1 + 1 = 4.$$

(b) (6 points) $\sum_{k=2}^{4}(k^2+4)$

Solution: Calculate

$$\sum_{k=2}^{4} k^2+4=(2^2+4)+(3^2+4)+(4^2+4)=8+13+20=41.$$

6. (28 points) In this problem, you will sketch $f(x)=2x+\frac{8}{x}$.

(a) (5 points) Compute f' and f''.

Solution: Calculate

$$f'(x)=2-\frac{8}{x^2}$$

and

$$f''(x)=\frac{16}{x^3}.$$

(b) (5 points) Find the critical point(s) and possible inflection point(s) of f.

Solution: To find critical points, solve $f'(x)\overset{\text{set}}{=}0$: $2x^2=8$ to get $x=\pm 2$. To find possible inflection points, solve $f''(x)\overset{\text{set}}{=}0$ to get $x=0$.

(c) (5 points) Find where f is increasing and decreasing and where it is concave up and concave down.

Solution: For increasing and decreasing, we will evaluate f' at the test points $x=-3$, $x=-1$, $x=1$, and $x=3$:

$$f'(-3)=2-\frac{8}{9}>0,$$
$$f'(-1)=2-8<0,$$
$$f'(1)=2-8<0,$$

and

$$f'(3)=2-\frac{8}{9}>0.$$

Therefore f is increasing on $(-\infty,-2)\cup(2,\infty)$ and f is decreasing on $(-2,2)$. For concavity, evaluate f'' at the test points $x=-1$ and $x=1$:

$$f''(-1)=\frac{16}{(-1)^3}<0$$

and

$$f''(1)=\frac{16}{1^3}>0,$$

therefore f is concave down on $(-\infty,0)$ and f is concave up on $(0,\infty)$.

(d) (6 points) Find the y-intercept, x-intercept(s), and any vertical asymptotes of f, if they exist.

Solution: Notice there is no y-intercept because $f(0)$ does not exist. Coincidentally, there is a vertical asymptote there. To find x-intercepts, solve $f(x)=0$:

$$2x+\frac{8}{x}=0,$$

so

$$2x^2+8=0,$$

or

$$x^2=-4,$$

which has no real solutions (its solutions are $\pm 2i$ — complex!). Therefore the graph has no x-intercept.

(e) (7 points) Use the information you found in parts (a)–(d) to sketch f. *Solution:*