MATH 1190 - EXAM 2 FALL 2016
SOLUTION

Wednesday 5 October 2016
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (10 points) Use the rules of differentiation to compute the derivative of...

(a) (2 points) $f(x) = 3x^4 + 2x^2 + 11x + 5$

Solution: Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}[3x^4 + 2x^2 + 11x + 5] = 12x^3 + 4x + 11.$$

(b) (2 points) $f(x) = x^2 \sin(x)$

Solution: Use the product rule to compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[x^2 \sin(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[x^2] \sin(x) + x^2 \frac{\mathrm{d}}{\mathrm{d}x}[\sin(x)] \\ &= 2x \sin(x) + x^2 \cos(x).\end{aligned}$$

(c) (2 points) $f(x) = \dfrac{x^3 + 2}{x^2 + 1}$

Solution: Use the quotient rule to compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^3 + 2}{x^2 + 1}\right] &= \frac{(x^2 + 1)\frac{\mathrm{d}}{\mathrm{d}x}[x^3 + 2] - (x^3 + 2)\frac{\mathrm{d}}{\mathrm{d}x}[x^2 + 1]}{(x^2 + 1)^2} \\ &= \frac{(x^2 + 1)(3x^2) - (x^3 + 2)(2x)}{(x^2 + 1)^2} \\ &= \frac{3x^4 + 3x^2 - (2x^4 + 4x)}{(x^2 + 1)^2} \\ &= \frac{x^4 + 3x^2 - 4x}{(x^2 + 1)^2}.\end{aligned}$$

(d) (2 points) $f(x) = (x^2 + 3)^{50}$

Solution: Use the chain rule to compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2 + 3)^{50}\right] &= 50(x^2 + 3)^{49} \frac{\mathrm{d}}{\mathrm{d}x}[x^2 + 3] \\ &= 100x(x^2 + 3)^{49}.\end{aligned}$$

(e) (2 points) $f(x) = \tan(\cos(x))$

Solution: Recall that $\frac{\mathrm{d}}{\mathrm{d}x} \tan(x) = \sec^2(x)$. Use the chain rule to compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\tan(\cos(x))] &= \sec^2(\cos(x)) \frac{\mathrm{d}}{\mathrm{d}x}[\cos(x)] \\ &= -\sin(x) \sec^2(\cos(x)).\end{aligned}$$

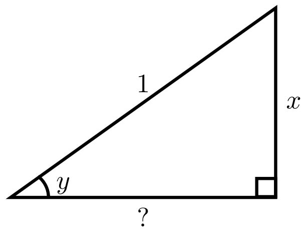
2. (10 points) Use implicit differentiation to determine the derivative of $f(x) = \arcsin(x)$, the inverse sine function. Express your final answer in terms of the variable x.
note: full credit will only be awarded for a correct process, not simply having the correct final answer.
Solution: Let $y = \arcsin(x)$ so that $\sin(y) = x$. We seek $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Now differentiate implicitly ot get

$$\cos(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos(y)}.$$

Since $\sin(y) = x$, we may draw the following triangle from right triangle trigonometry:



To find the value of ?, use the Pythagorean theorem to see

$$?^2 + x^2 = 1^2,$$

or

$$? = \pm\sqrt{1-x^2}.$$

We drop the negative solution (it is not meaningful here) and we get $? = \sqrt{1-x^2}$. Therefore we may compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}.$$

3. (10 points) Consider y to be a function of x. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, where

$$x^4 - x^2y + y^3 = 7x.$$

Solution: Differentiate implicitly to get

$$4x^3 - 2xy - x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 7.$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left[3y^2 - x^2\right] = 7 - 4x^3 + 2xy.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{7 - 4x^3 + 2xy}{3y^2 - x^2}.$$

4. (10 points) Find an equation of the tangent line to the "Conchoid of de Sluze"

$$x^3 - x^2 + xy^2 - y^2 = -4x^2$$

at the point $(-1, 1)$.
Solution: First differentiate implicitly to get

$$3x^2 - 2x + y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = -8x.$$

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-6x - 3x^2 - y^2}{2xy - 2y}.$$

Now plug in the point $(-1, 1)$ (i.e. $x = -1$ and $y = 1$) to get the slope of the tangent line at $(-1, 1)$:

$$\text{slope} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-6(-1) - 3(-1)^2 - (-1)^2}{2(-1)(1) - 2(1)} = \frac{6 - 3 - 1}{-4} = \frac{2}{-4} = -\frac{1}{2}.$$

Now use the point-slope form of the equation of line, (recall: $y - y_1 = m(x - x_1)$) to get the equation of the tangent line:

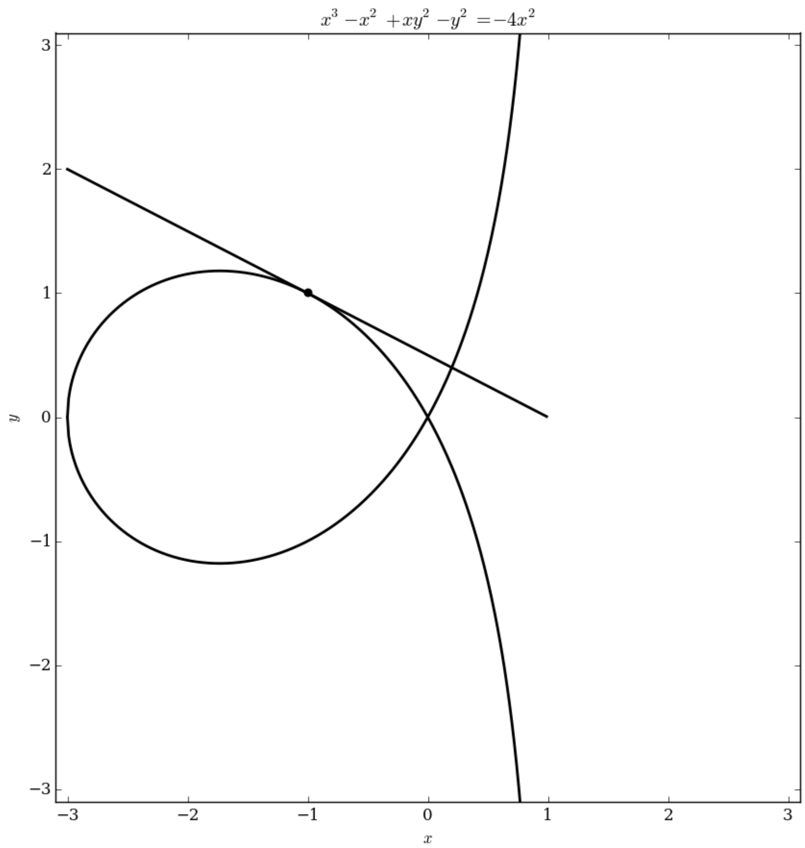
$$y - 1 = -\frac{1}{2}(x - (-1)),$$

or

$$y - 1 = -\frac{1}{2}(x + 1),$$

or

$$y = -\frac{1}{2}x + \frac{1}{2}.$$



5. (10 points) The area of a circle is increasing at a rate of $5\,\frac{\text{cm}^2}{\text{min}}$. Find the rate of change of the radius when the radius is 1 cm.

Solution:

Variables: $A \sim$ area, $r \sim$ radius

Known Rate: $\frac{\mathrm{d}A}{\mathrm{d}t} = 5\frac{\text{cm}^2}{\text{min}}$

Equation: $A = \pi r^2$

We seek: $\frac{\mathrm{d}r}{\mathrm{d}t}$ when $r = 1$ cm.

Differentiate the equation implicitly to get

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}.$$

Now plug in the known rate $\frac{\mathrm{d}A}{\mathrm{d}t} = 5$ and $r = 1$ to get

$$5 = 2\pi(1)\frac{\mathrm{d}r}{\mathrm{d}t}.$$

Solve for $\frac{\mathrm{d}r}{\mathrm{d}t}$ to get

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{2\pi}\frac{\text{cm}}{\text{min}}.$$

6. (10 points) The "ideal gas law" in chemistry says that the pressure P, volume V, and temperature T of a certain closed system is

$$PV = 5T.$$

If the pressure is increasing at a rate of $3\frac{\text{Pa}}{\text{sec}}$ and the temperature is increasing at a rate of $2\frac{^\circ\text{F}}{\text{sec}}$, at what rate is the volume changing when the pressure is 2 Pa and the volume is 100cm^3?

Solution:

Variables: $P \sim$ pressure, $V \sim$ volume, $T \sim$ temperature

Known Rates: $\frac{\mathrm{d}P}{\mathrm{d}t} = 3\frac{\text{Pa}}{\text{sec}}$ and $\frac{\mathrm{d}T}{\mathrm{d}t} = 2\frac{^\circ\text{F}}{\text{sec}}$

Equation: $PV = 5T$

We seek: $\frac{\mathrm{d}V}{\mathrm{d}t}$ when $P = 2$ and $V = 100$

Differentiate the equation implicitly to get

$$P\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}P}{\mathrm{d}t}V = 5\frac{\mathrm{d}T}{\mathrm{d}t}.$$

Plug in the known rates, $P = 2$, and $V = 100$ to get

$$2\frac{\mathrm{d}V}{\mathrm{d}t} + 3(100) = 5(2).$$

Now solve for $\frac{\mathrm{d}V}{\mathrm{d}t}$ to get

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{10-300}{2} = \frac{-290}{2} = -145\frac{\text{cm}^3}{\text{sec}}.$$

7. (10 points) Find the critical numbers of the function

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - x.$$

Solution: Differentiate and set the derivative equal to zero to get

$$f'(x) = x^2 + x - 1 \overset{\text{set}}{=} 0.$$

Now solve using the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

Therefore the critical numbers are $-\frac{1}{2} + \frac{\sqrt{5}}{2}$ and $-\frac{1}{2} - \frac{\sqrt{5}}{2}$.

8. (10 points) Find the absolute extrema of $f(x) = x^2 - \frac{3}{2}x$ on the interval $[-1, 2]$.
Solution: First find the critical numbers of f. Differentiate and set equal to zero to get

$$f'(x) = 2x - \frac{3}{2} \stackrel{\text{set}}{=} 0.$$

Solve this equation to get $x = \frac{1}{2}\frac{3}{2} = \frac{3}{4}$. This critical number lies in the interval $[-1, 2]$, so we do not throw it out. Now we compare the value of f at the critical number and on the endpoints of the interval $[-1, 2]$:

$$f(-1) = (-1)^2 - \frac{3}{2}(-1) = 1 + \frac{3}{2} = \frac{5}{2},$$

$$f(2) = 2^2 - 2\frac{3}{2} = 4 - 3 = 1,$$

and

$$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^2 - \frac{3}{2}\left(\frac{3}{4}\right) = \frac{9}{16} - \frac{9}{8} = \frac{9}{16} - \frac{18}{16} = -\frac{9}{16}.$$

Therefore the absolute maximum occurs at $x = -1$ with value $\frac{5}{2}$ and the absolute minimum occurs at $x = \frac{3}{4}$ with value $-\frac{9}{16}$.

9. (10 points) Determine if Rolle's theorem can be applied to the function $f(x) = \sin(x)$ on the interval $[0, 2\pi]$. If it can, find all values of c in the open interval $(0, 2\pi)$ such that $f'(c) = 0$.
Solution: Rolle's theorem does apply because $\sin(x)$ is differentiable is continuous and differentiable on the whole real line (hence also on $[0, 2\pi]$). To find the requested values of c, differentiate f and set it equal to zero to get $f'(x) = \cos(x) \stackrel{\text{set}}{=} 0$. This equation has most general solutions, for any integer k, $x = \frac{\pi}{2} + 2\pi k$ and $x = \frac{3\pi}{2} + 2\pi k$. However the only solutions that lie in the interval $[0, 2\pi]$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

10. (10 points) Determine if the Mean Value Theorem can be applied to the function $f(x) = \frac{x+1}{x}$ on the interval $[-1, 2]$.
If it can, find all values of c in the open interval $(-1, 2)$ such that $f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$. If it cannot, explain why not.
Solution: First, $x = 0$ lies in the interval $[-1, 2]$. Now notice that the function $\frac{x+1}{x}$ is not defined at $x = 0$ because it would cause a division by zero. This means that f is not continuous on the interval $[-1, 2]$, and so the mean value theorem does not apply.