MATH 3304 - EXAM 2 SUMMER 2015
SOLUTIONS

Thursday 2 July 2015
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (29 points) Use the **method of undetermined coefficients** to solve the following differential equation:

$$y'' - 5y' - 6y = 3e^{6t}.$$

Solution: Solve the homogeneous equation $y'' - 5y' - 6y = 0$ by considering the characteristic equation $r^2 - 5r - 6 = 0$. It factors to $(r-6)(r+1) = 0$ and so we have the homogeneous solution

$$y_h(t) = c_1 e^{-t} + c_2 e^{6t}.$$

The method of undetermined coefficients tells us to make the guess

$$y_p(t) = t^s(Ae^{6t}).$$

Using the notation from the method of undetermined coefficients table, we have $\alpha = 6$ and $\beta = 0$. We ask: what is the multiplicity of $\alpha + \beta i = 6$ as a root of the characteristic equation? Since the root $r = 6$ appears only once, we get $s = 1$. Thus our guess can be refined to

$$y_p(t) = Ate^{6t}.$$

To find A we first compute $y_p'(t) = (A + 6At)e^{6t}$ and we compute $y_p''(t) = (12A + 36At)e^{6t}$. Now plug these into the differential equation to get

$$e^{6t}\left[(12A + 36At) - 5(A + 6At) - 6(At)\right] = 3e^{6t}.$$

which simplifies to $7A = 3$, or $A = \frac{3}{7}$. We have found the particular solution $y_p(t) = \frac{3}{7}te^{6t}$. Therefore the general solution of the differential equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-t} + c_2 e^{6t} + \frac{3}{7}te^{6t}.$$

2. (28 points) Find the general solution of the following differential equation:

$$y'' + y = \csc(t); 0 < t < \pi.$$

Solution: This problem cannot be solved using the method of undetermined coefficients, so we will use the method of variation of parameters. First we solve the homogeneous equation $y'' + y = 0$ which has characteristic equation $r^2 + 1 = 0$ which has roots $r = \pm i$. Thus the solution of the homogeneous equation is

$$y_h(t) = c_1 \cos(t) + c_2 \sin(t).$$

Compute the Wronskian

$$W\{y_1, y_2\}(t) = \det \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} = \cos^2(t) - (-\sin^2(t)) = 1.$$

We now "guess" the particular solution to be

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t)\cos(t) + u_2(t)\sin(t).$$

Variation of parameters allows us to compute u_1 and u_2 as follows:

$$u_1 = -\int \frac{\csc(t)\sin(t)}{1} dt = -\int 1 dt = -t$$

and

$$u_2 = \int \frac{\csc(t)\cos(t)}{1} dt = \int \frac{\cos(t)}{\sin(t)} dt = \log|\sin(t)|.$$

Therefore our particular solution is

$$y_p(t) = -t\cos(t) + \log(|\sin(t)|)\sin(t),$$

and the general solution is

$$y(t) = c_1 \cos(t) + c_2 \sin(t) - t\cos(t) + \log(|\sin(t)|)\sin(t).$$

3. (28 points) Use reduction of order to find the general solution of

$$t^2y''(t) + 2ty'(t) - 2y(t) = 0$$

given that $y_1(t) = t$ is a solution and $t > 0$.
Solution: Reduction of order tells us to "guess" $y_2(t)$ to be of the form $y_2(t) = v(t)y_1(t) = v(t)t$. From this we can compute $y_2'(t) = v'(t)t + v(t)$ and $y_2''(t) = v''(t)t + 2v'(t)$. Plugging these into the original equation yields

$$t^2(v''t + 2v') + 2t(v't + v) - 2vt = 0,$$

and upon simplification we get

$$tv'' + 4v' = 0,$$

which can be solved as a first order problem by writing $w = v'$ to get

$$tw' + 4w = 0.$$

This first order problem can be solved using separation of variables:

$$\int \frac{1}{w} dw = -\int \frac{4}{t} dt$$

yielding the solution (for some constant A),

$$w = \frac{A}{t^4}.$$

We may solve for v by integration: $v' = w$ implies $v = \int w$ and so

$$v(t) = \int \frac{A}{t^4} dt = \frac{C}{t^3},$$

for some constant C. We have found the solution $y_2(t) = vt = \frac{C}{t^2}$. Therefore the general solution is

$$y(t) = c_1 t + c_2 \frac{1}{t^2}.$$

4. (28 points) Find the general solution of the following differential equations:

(a) (23 points) $y'''(t) + y''(t) + 81y'(t) + 81y(t) = 0$
Solution: The characteristic equation for this problem is

$$r^3 + r^2 + 81r + 81 = 0.$$

The left-hand-side factors to $(r^2 + 81)(r - 1) = 0$, yielding roots $r = 1, \pm 9i$. Hence the general solution is

$$y(t) = c_1 e^t + c_2 \cos(9t) + c_2 \sin(9t).$$

(b) (5 points) $y'''(t) = t^2 + 1$
Solution: This problem can be solved by integration. Integrate once to get

$$y''(t) = \frac{t^3}{3} + t + c_1,$$

integrate again to get

$$y'(t) = \frac{t^4}{12} + \frac{t^2}{2} + c_1 t + c_2,$$

and integrate once more to get

$$y(t) = \frac{t^5}{60} + \frac{t^3}{6} + c_1 \frac{t^2}{2} + c_2 t + c_3$$

or alternatively using $\tilde{c_1} = 2c_1$

$$y(t) = \frac{t^5}{60} + \frac{t^3}{6} + \tilde{c_1} t^2 + c_2 t + c_3.$$

5. (28 points) Use $g = 32\frac{\text{ft}}{\text{s}^2}$. A spring hangs vertically from a rigid support. When a body with mass $\frac{1}{16}$slug is attached to the spring, it stretches $\frac{1}{4}$ft. The body is given a downward displacement of $\frac{1}{2}$ft and released with no initial velocity. Assume there there is no damping force and no forcing function.

(a) (25 points) Determine the position of the body at time t.
Solution: We compare the given information to the general equation for spring problems:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t).$$

We are told that $m = \frac{1}{16}$, $\gamma = 0$, and $F(t) = 0$. We must find the value of k. We are told $u_0 = \frac{1}{4}$ and so we arrange the equation $mg = u_0 k$ to get

$$k = \frac{mg}{u_0} = \frac{\frac{32}{16}}{\frac{1}{4}} = 8.$$

The initial conditions are $u(0) = \frac{1}{2}$ and $u'(0) = 0$. Thus we have the initial value problem

$$\frac{1}{16}u'' + 8u = 0; u(0) = \frac{1}{2}, u'(0) = 0.$$

To solve it, find the solution by solving the characteristic equation $\frac{1}{16}r^2 + 8 = 0$ which has solution $r = \pm\sqrt{-128} = \pm 8\sqrt{2}i$ yielding the solution

$$u(t) = c_1 \cos(8\sqrt{2}t) + c_2 \sin(8\sqrt{2}t).$$

We now need to find the values of c_1 and c_2 from the initial conditions. Calculate

$$u'(t) = -8\sqrt{2}c_1 \cos(8\sqrt{2}t) + 8\sqrt{2}c_2 \sin(8\sqrt{2}t).$$

Thus our initial conditions are given by

$$\begin{cases} \frac{1}{2} = u(0) = c_1 \\ 0 = u'(0) = -8\sqrt{2}c_2 \end{cases}$$

and so we see $c_1 = \frac{1}{2}$ and $c_2 = 0$. Therefore the solution of the initial value problem is

$$u(t) = \frac{1}{2}\cos(8\sqrt{2}t).$$

(b) (3 points) At which time $t > 0$ does the body in the system described above first return to its equilibrium position?
Solution: The body will return to equilibrium for the first time at the smallest value of t such that $u(t) = \frac{1}{2}\cos(8\sqrt{2}t) = 0$. This occurs whenever the argument to the cosine function is $\frac{\pi}{2}$, i.e. $8\sqrt{2}t = \frac{\pi}{2}$. Therefore the body returns to equilibrium the first time when $t = \frac{\pi}{16\sqrt{2}}$.

2015 Summer Semester MATH 3304 Hour Exam 2
Instructor: Tom Cuchta, Section A

Points earned (out of 140)	How many got this score?
num	num

Number taking exam:
Median: points ()
Mean: points ()
Standard deviation: points ()
Number receiving A's ($126 \leq$ points ≤ 140):
Number receiving B's ($112 \leq$ points < 126):
Number receiving C's ($98 \leq$ points < 112):
Number receiving D's ($84 \leq$ points < 98):
Number receiving F's ($0 \leq$ points < 84):