The Structure of Zero-Divisors

Tom Cuchta

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- Both operations are associative, i.e. $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Both operations are commutative, i.e. $a+b = b+a$ and $a \cdot b = b \cdot a$.

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- For every element a, there is an element b such that $a + b = 0$. We denote this element by $-a$, and call it the additive inverse of a.
- Multiplication distributes over addition, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$.

Examples of rings

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- The integers modulo n, $\mathbb{Z}_n$ is a ring under addition modulo n and multiplication modulo n. For example, "clock arithmetic" where 8 o'clock plus 6 hours is 2 o'clock. "Clock arithmetic" is $\mathbb{Z}_{12}$.

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- $\mathbb{Z}[x]$, polynomials with coefficients from $\mathbb{Z}$.
- $\mathbb{R}[[x]]$, power series with real coefficients.

Ideals

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- An *ideal* of a ring is a subring such that the product of any ring element with an element of the ideal is in the ideal.
- A *maximal ideal* is an ideal that is not contained in any other ideal besides the entire ring.
- A *local ring* is a ring with exactly one maximal ideal.

Special ring elements

- The *zero-divisors* of a ring are the elements z such that $z \cdot r = 0$ for some nonzero r.
  For example, 0 is always a zero-divisor. In $\mathbb{Z}$ there are not any other zero-divisors. In $\mathbb{Z}_{12}$, both 3 and 4 are zero-divisors, since $3 \cdot 4 = 12 \equiv 0$.

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- The *nilpotents* of a ring are the elements r such that $r^n = 0$ for some positive integer n.
  Note that nilpotents are always zero-divisors.

Algebraic structure of zero-divisors

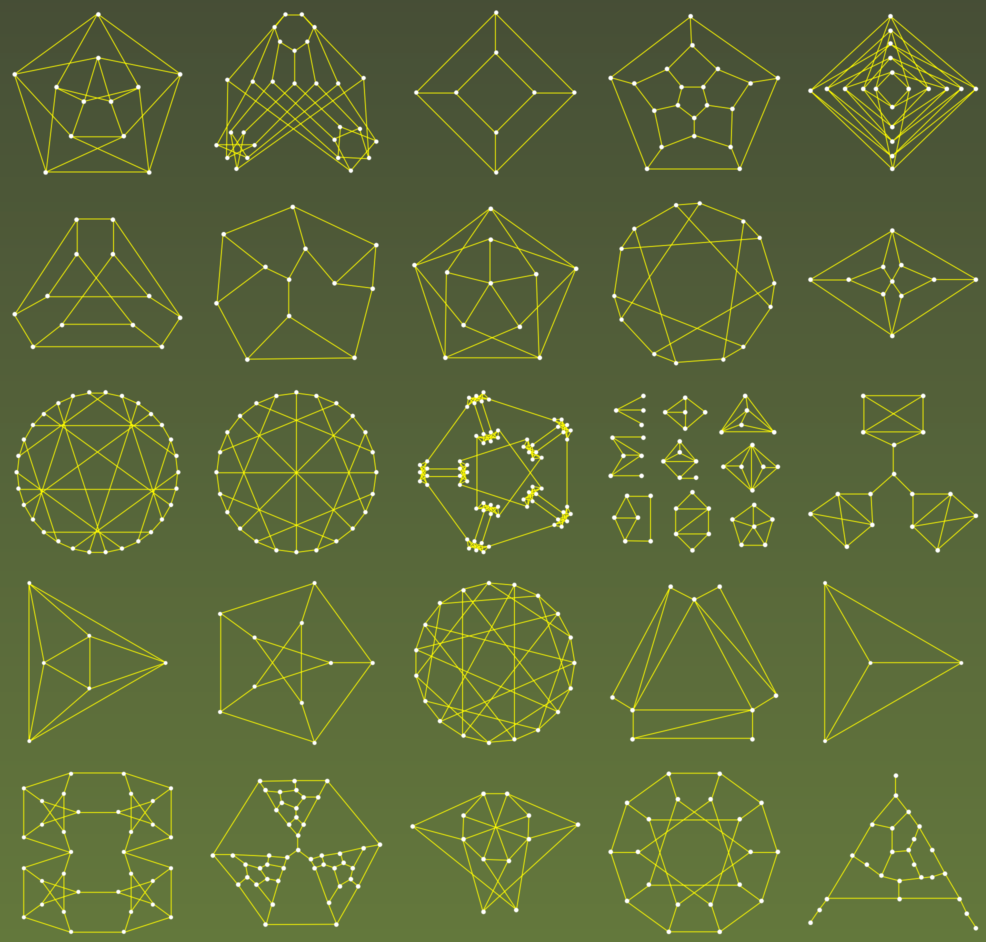
Theorem: (Axtell, Stickles, Trampbachls)
Z(R) is an ideal if and only if R is local

This theorem shows that $Z(R)$ has the nice structure of an ideal under *very restrictive* circumstances.

It would be nice to have a more general invariant that indicates the structure of zero-divisors. Luckily we have such a thing!

Introduction to graphs

A *graph* is a set of vertices and edges connecting some of the vertices.



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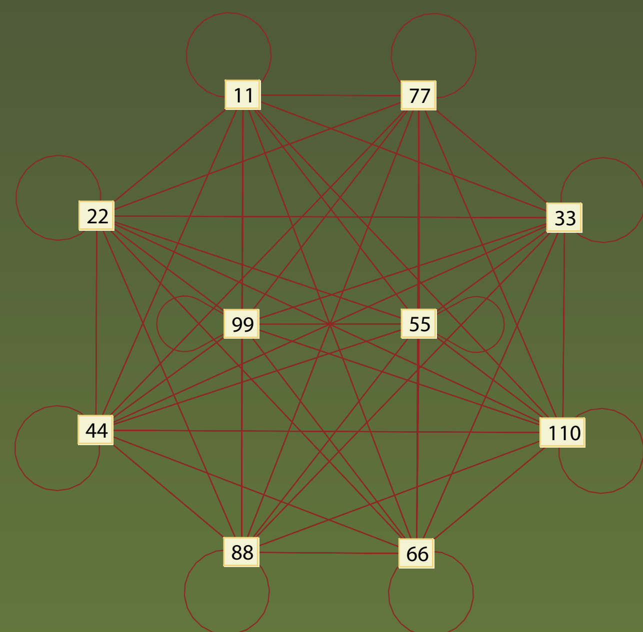
- The *diameter* of a graph is the maximal distance between all pairs of vertices.
- A *cycle* is a path from a vertex to itself that does not repeat edges.
- The *girth* of a graph is the length of the smallest cycle (ignoring loops). If there are no cycles, we say the girth is ∞.

Zero-divisor graphs

The *zero-divisor graph*, $\Gamma(R)$, of a commutative ring R is the graph whose vertices are the nonzero zero-divisors of R and two vertices are connected by an edge if and only if their product is 0.



(a) $N(\mathbb{Z}_{121})$

Zero-divisor graphs

Theorem: (D.F. Anderson and P.S. Livingston, 1999) (slightly improved) The zero-divisor graph of a commutative ring is connected with diameter ≤ 3 and girth 3, 4, or ∞.

This is quite a structure! This theorem applies to any commutative ring while the previous algebraic theorem applied only to local rings!

Zero-divisor graphs

Theorem: (T. Cuchta, K. Lokken, W. Young, 2008)
The following table holds true:

Factorization of n	Diameter	Girth
p; p is prime	-	-
2^2	0	∞
3^2	1	∞
p^2; p is prime and $p > 3$	1	3
2^3, or $2p$; p odd prime	2	∞
pq; p,q, distinct odd primes	2	4
p^m; p is prime, $m > 2$, and $p^m \neq 8$	2	3
$4p$; p is an odd prime	3	4
pqk; p, q distinct primes, $k \in \mathbb{Z}^+$ and pqk does not meet any criteria listed above	3	3

Nilradical and non-nilradical graphs

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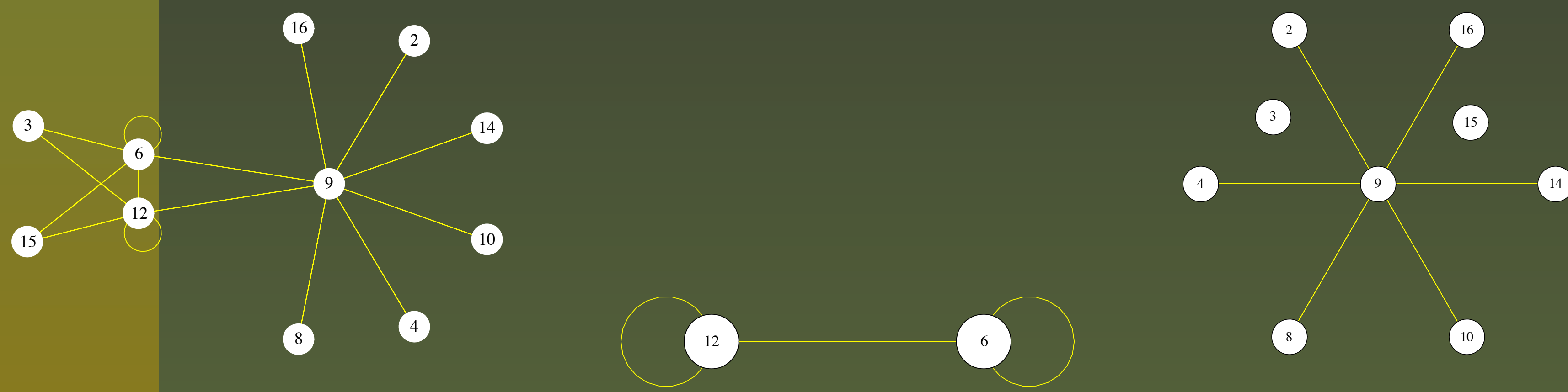
- We define the *non-nilradical graph*, denoted $\Omega(R)$, to be the graph whose vertices are the non-nilpotent zero-divisors of R and two vertices are connected by an edge if and only if their product is 0.

Pictures of $\mathbb{Z}_{18}$



(b) The zero-divisor graph, $\Gamma(\mathbb{Z}_{18})$

(c) $N(\mathbb{Z}_{18})$

(d) $\Omega(\mathbb{Z}_{18})$

Figure 1: The three graphs of $\mathbb{Z}_{18}$

Nilradical graphs are connected too!

Theorem: (A. Bishop, T. Cuchta, K. Lokken, O. Pechenik, 2008)
$N(R)$ is connected with diameter ≤ 2 and girth 3 or ∞.

This theorem is not too surprising because it is well known that $nil(R)$ is an ideal of R, and thus a ring. However, notice that the diameter and girth are more restricted in $N(R)$ than in $\Gamma(R)$.

More definitions

- An *isolated vertex* is a vertex that has no incident edges.

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- A graph is *almost connected* if there exists a path between any two non-isolated vertices.

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- **Theorem: (A. Bishop, T. Cuchta, K. Lokken, O. Pechenik, 2008)**
  $\Omega(R)$ is almost connected and the connected component has diameter ≤ 3 and girth $3, 4$, or ∞.

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- This theorem is quite surprising! The diameter and girth restrictions on $\Omega(R)$ are the same as the restrictions on general zero-divisor graphs!

- Moreover, this theorem defies all intuition. We removed the nilradical, a highly structured subring of R, from the zero-divisors, an algebraic set with very little general structure, and the resultant set is still graphically structured!

$N(\mathbb{Z}_n)$ by diameter and girth

Theorem: (A. Bishop, T. Cuchta, K. Lokken, O. Pechenik, 2008)

Factorization of n	Diameter	Girth
$p_1 p_2 \ldots p_m$ such that all p_i are distinct primes	-	-
$4k$, $\gcd(2, k) = 1$, $p^2 \nmid k$, for all prime p	0	∞
$9k$, $\gcd(3, k) = 1$, $p^2 \nmid k$, for all prime p	1	∞
p^2, p prime, $p > 3$	1	3
$2p^2$, p prime, $p > 3$	1	3
$p^2 q^2$, p and q prime, $p \neq q$	1	3
$p^2 d$, $\gcd(p, d) = 1$, p prime, $p > 3$, d not divisible by any non-trivial cube	1	3
$8k$, $\gcd(8, k) = 1$, $p^2 \nmid k$, for all prime p	2	∞
$p^\ell a$, $\ell \geq 3$, p prime, $p > 2$	2	3
$2^\ell b$, $\ell \geq 3$, b not a product of distinct primes	2	3

Ω_c by diameter and girth

Theorem: (A. Bishop, T. Cuchta, K. Lokken, O. Pechenik, 2008)

Factorization of n	Diameter	Girth
p^m, where p is prime and $m \in (Z)^+$	-	-
$2p^k$, where p is an odd prime and $k > 1$	2	∞
$p^k q^\ell$, where p, q are distinct primes, k, $\ell \in \mathbb{Z}^+$, and $p^k, q^\ell \neq 2$	2	4
$p_1^{e_1} p_2^{e_2} \ldots p_c^{e_c}$, all p_i are distinct, $e_i \in \mathbb{Z}^+$, and $c \geq 3$	3	3

This is the first table we proved. Eventually, we learned a little about Artinian rings and managed to generalize it to all finite commutative rings!

Ω_c by diameter and girth

Theorem: (A. Bishop, T. Cuchta, K. Lokken, O. Pechenik, 2008)

R	Diameter	Girth
R is local	-	-
$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	2	∞
$R \cong L_1 \times L_2$	2	4
$R \cong L_1 \times L_2 \times \ldots \times L_n, n > 2$	3	3

This is the major result in the paper. This table holds true for any Artinian ring (ring that meets the descending chain condition on ideals) because any Artinian ring can be broken into a finite direct product of local rings (fields are local rings! The maximal ideal in a field is 0.)

A Potpourri of Properties of $\Omega(R)$

- For R, a finite commutative ring with unity, $\Omega(R)$ contains at least one isolated vertex if and only if $\text{nil}(R) \neq \{0\}$ and $\text{nil}(R) \neq Z(R)$.

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- For R, a finite commutative ring with unity, $\Omega(R)$ contains at least one isolated vertex if and only if $\text{nil}(R) \neq \{0\}$ and $\text{nil}(R) \neq Z(R)$.

- Let R be a finite commutative ring with unity, $\Omega_c(R) \cup \{0\}$ is multiplicatively closed, unless R is isomorphic to the cross product of three or more local rings, $L_1, L_2, L_3, \ldots L_n$, and $\Gamma(L_i)$ is not complete for some i.

A Potpourri of Properties of $\Omega(R)$

- Any finite ring R can be decomposed into a finite direct product of local rings.

A Potpourri of Properties of $\Omega(R)$

- Any finite ring R can be decomposed into a finite direct product of local rings.
- Let R be a non-local finite commutative ring with unity. Then, $\chi(\Omega(R)) = n$, where n is the number of local rings in the decomposition of R.

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