Laplace Transforms on Time Scales

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Overview

1. Introduction

2. Exponential Functions

3. The Laplace Transform

A time scale is a set $\mathbb{T} \subset \mathbb{R}$ closed under the standard topology on $\mathbb{R}$. Define the forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{z \in \mathbb{T}\colon z > t\}$$

the graininess $\mu\colon \mathbb{T} \to \mathbb{R}$ by

$$\mu(t) = \sigma(t) - t$$

and the minimal graininess function $\mu_*\colon \mathbb{T} \to \mathbb{R}$ by

$$\mu_*(s) = \inf_{\tau \in [s,\infty)\cap\mathbb{T}} \mu(\tau).$$

A function $f\colon \mathbb{T} \to \mathbb{C}$ is called Δ-differentiable at $t \in \mathbb{T}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$ and there exists a number $f^{\Delta}(t)$ such that

$$|[f(\sigma(s)) - f(s)] - f^{\Delta}(s)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|.$$

Integration is defined so that $\displaystyle\int_s^t f^{\Delta}(\tau)\Delta\tau = f(t) - f(s)$.

If $\mathbb{T}$ consists of just isolated points, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

and

$$\int_a^b f(t)\Delta t = \begin{cases} \displaystyle\sum_{t \in [a,b)\cap\mathbb{T}} \mu(t)f(t) & ; a < b \\ 0 & ; a = b \\ \displaystyle -\sum_{t \in [b,a)\cap\mathbb{T}} \mu(t)f(t) & ; a > b. \end{cases}$$

Note: $\tilde{q} := q-1$, $\mathbb{Z}^2 := \{n^2 : n \in \mathbb{Z}\}$

Time Scale	$\sigma(t) =$	$\mu(t) =$	$f^{\Delta}(t) =$	$\int_s^t f(\tau)\Delta\tau =$
$\mathbb{R}$	t	0	$\lim\limits_{h\to 0} \dfrac{f(t+h)-f(t)}{h}$	$\int_s^t f(\tau)d\tau$
$h\mathbb{Z}$	$t+h$	h	$\dfrac{f(t+h)-f(t)}{h}$	$\sum\limits_{k=\frac{s}{h}}^{\frac{t}{h}} hf(hk)$
$q^{\mathbb{N}}; q>1$	qt	$\tilde{q}t$	$\dfrac{f(qt)-f(t)}{t\tilde{q}}$	$\tilde{q} \sum\limits_{k=\log_q(s)}^{\log_q(t)-1} q^k f(q^k)$
$\mathbb{Z}^2$	$(1+\sqrt{t})^2$	$2\sqrt{t}+1$	$\dfrac{f((1+\sqrt{t})^2)-f(t)}{2\sqrt{t}-1}$	$\sum\limits_{k=\sqrt{s}}^{\sqrt{t}-1} (2k+1)f(k^2)$

We will need some modified sets related to $\mathbb{C}$. First the Hilger complex plane is the set

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} \colon z \neq -\frac{1}{h} \right\}$$

where we take $\mathbb{C}_0 := \mathbb{C}$. The Hilger real part of a complex number z is defined to be

$$\mathrm{Re}_h(z) := \frac{1}{h}(|1 + hz| - 1);$$

it is known that if $h \geq 0$ then $\mathrm{Re}_h(z)$ is an increasing function of h. Define

$$\mathbb{C}_h(\lambda) := \{ z \in \mathbb{C}_h \colon \mathrm{Re}_h(z) > \lambda \}.$$

A function $p\colon \mathbb{T} \to \mathbb{C}$ is called regressive if for all $t \in \mathbb{T}$,

$$1 + \mu(t)p(t) \neq 0.$$

We denote the set of regressive functions with domain X and codomain Y as $\mathscr{R}(X, Y)$. If $1 + \mu(t)p(t) > 0$ for all t, then we say that p is positively regressive and that $p \in \mathscr{R}^+(\mathbb{T}, X)$. We use the notation $\mathscr{R}_c$ to denote constant regressive functions.
Let p be regressive. The exponential function $e_p\colon \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ is defined as

$$e_p(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\}$$

where

$$\xi_h(z) := \frac{1}{h} Log(1 + zh).$$

The solution of the IVP

$$y^\Delta = py; y(s) = 1$$

is $e_p(\cdot, s)$.

Time Scale	Exponential Function $e_p(t,s) =$
$\mathbb{R}$	$\exp\left\{\int_s^t p(\tau)d\tau\right\}$
$h\mathbb{Z}$	$\prod_{k=\frac{s}{h}}^{\frac{t}{h}-1} 1+hp(hk)$
$q^{\mathbb{N}}; q>1$	$\prod_{k=\log_q(s)}^{\log_q(t)-1} 1+p(q^k)q^k(q-1)$
$\mathbb{Z}^2$	$\prod_{k=\sqrt{s}}^{\sqrt{t}-1} 1+p(k^2)(2k+1)$

Let $f, g \in \mathscr{R}(\mathbb{T}, \mathbb{C})$. Define the operations $\oplus, \ominus$ by the formulas

$$(f \oplus g)(t) := f(t) + g(t) + \mu(t)f(t)g(t)$$

and

$$(f \ominus g)(t) := \frac{f(t) - g(t)}{1 + \mu(t)g(t)}.$$

Then

- $e_p(s, s) = 1$
- $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$
- $\dfrac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$
- $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$

Let $\lambda \in \mathscr{R}_c^+([s,\infty)\cap\mathbb{T},\mathbb{R})$. Define

$$m_\lambda(t,s) := \int_s^t \frac{1}{1+\mu(\tau)\lambda}\Delta\tau.$$

Theorem

(Decay of Exponential Function) Let $\mathbb{T}$ be a time scale such that $\sup\mathbb{T}=\infty$ and let $s \in \mathbb{T}$ and $\lambda \in \mathscr{R}_c^+([s,\infty)\cap\mathbb{T},\mathbb{R})$. Then for any $z \in \mathbb{C}_{\mu_*(s)}(\lambda)$,

1. $|e_{\lambda\ominus z}(t,s)| \leq e_{\lambda\ominus \mathrm{Re}_{\mu_*(s)}(z)}(t,s)$
2. $\lim\limits_{t\to\infty} e_{\lambda\ominus \mathrm{Re}_{\mu_*(s)}(z)}(t,s) = 0$
3. $\lim\limits_{t\to\infty} e_{\lambda\ominus z}(t,s) = 0.$

Corollary

For $z \in \mathbb{C}_{\mu_*(s)}(\lambda)$,

$$e_{\lambda\ominus \mathrm{Re}_{\mu_*(s)}(z)}(t,s) \leq \exp\left\{-\left[\mathrm{Re}_{\mu_*(s)}(z)-\lambda\right] m_{\mathrm{Re}_{\mu_*(s)}(z)}(t,s)\right\}$$

A function $f \in C_{rd}(\mathbb{T}, \mathbb{C})$ has exponential order α on $[s, \infty) \cap \mathbb{T}$ if $\alpha \in \mathscr{R}_c^+([s, \infty) \cap \mathbb{T}, \mathbb{R})$ and there exists $K > 0$ such that $|f(t)| \leq K e_\alpha(t, s)$ for all $t \in [s, \infty) \cap \mathbb{T}$.

For some $f \in C_{rd}(\mathbb{T}, \mathbb{C})$ of exponential order α define the Laplace transform of f about s by the formula

$$\mathscr{L}\{f\}(z; s) := \int_s^\infty f(\tau) e_{\ominus z}(\sigma(\tau), s) \Delta\tau.$$

Theorem

Let $f \in C_{rd}([s,\infty)\cap\mathbb{T},\mathbb{C})$ be of exponential order α. Then $\mathscr{L}\{f\}(\cdot;s)$ exists on $\mathbb{C}_{\mu_*(s)}(\alpha)$ and converges absolutely.

Proof.

Recall that $\mathrm{Re}_h(z)$ is an nondecreasing function of $h \geq 0$. Therefore

$$\mathrm{Re}_{\mu(t)}(z) \geq \mathrm{Re}_{\mu_*(s)}(z)$$

for all $t \in [s,\infty)\cap\mathbb{T}$. Thus

$$|1+\mu(t)z| \geq 1+\mu(t)\mathrm{Re}_{\mu_*(s)}(z).$$

Let $t \in [s,\infty)\cap\mathbb{T}$ and $z \in \mathbb{C}_{\mu_*(s)}$. Compute

$$e_{\ominus z}(\sigma(t),s) = \left(1 - \frac{z\mu(t)}{1+\mu(t)z}\right) e_{\ominus z}(t,s) = \frac{e_{\ominus z}(t,s)}{1+\mu(t)z}.$$

Proof.

Now using the Decay Theorem and that f is of exponential order α compute

$$\begin{aligned}
&\left| \int_s^t f(\tau) e_{\ominus z}(\sigma(\tau), s)\Delta\tau \right| \leq \int_s^t |f(\tau) e_{\ominus z}(\sigma(\tau), s)|\Delta\tau \\
&\leq K \int_s^t |e_\alpha(t,s) e_{\ominus z}(\sigma(t), s)|\Delta\tau \\
&= K \int_s^t \frac{|e_{\alpha \ominus z}(\tau, s)|}{|1+\mu(\tau)z|}\Delta\tau \\
&\leq K \int_s^t \frac{e_{\alpha \ominus \mathrm{Re}_{\mu_*(s)}(z)}(\tau, s)}{1+\mu(\tau)\mathrm{Re}_{\mu_*(s)}(z)}\Delta\tau \\
&= \frac{K}{\alpha - \mathrm{Re}_{\mu_*(s)}(z)} \int_s^t \frac{\alpha - \mathrm{Re}_{\mu_*(s)}(z)}{1+\mu(\tau)\mathrm{Re}_{\mu_*(s)}(z)} e_{\alpha \ominus \mathrm{Re}_{\mu_*(s)}(z)}(\tau, s)\Delta\tau
\end{aligned}$$

Proof.

$$\left| \int_s^t f(\tau) e_{\ominus z}(\sigma(\tau), s)\Delta\tau \right| \leq \frac{K}{\alpha - \mathrm{Re}_{\mu_*(s)}(z)} \int_s^t \frac{\alpha - \mathrm{Re}_{\mu_*(s)}(z)}{1+\mu(\tau)\mathrm{Re}_{\mu_*(s)}(z)} e_{\alpha\ominus \mathrm{Re}_{\mu*(s}}$$

$$= \frac{K}{\alpha - \mathrm{Re}_{\mu_*(s)}(z)} \int_s^t (\alpha \ominus \mathrm{Re}_{\mu_*(s)}(z)) e_{\alpha\ominus\mathrm{Re}_{\mu*(s)}(z)}(\tau, s)\Delta\tau$$

$$= \frac{K}{\alpha - \mathrm{Re}_{\mu_*(s)}(z)} \int_s^t e^{\Delta}_{\alpha\ominus\mathrm{Re}_{\mu*(s)}(z)}(\tau, s)\Delta\tau$$

$$= \frac{K}{\mathrm{Re}_{\mu_*(s)}(z) - \alpha} \left(1 - e_{\alpha\ominus\mathrm{Re}_{\mu*(s)}(z)}(t, s)\right)$$

Proof.

We have shown

$$\left|\int_s^t f(\tau)e_{\ominus z}(\sigma(\tau),s)\Delta\tau\right| \leq \frac{K}{\mathrm{Re}_{\mu_*(s)}(z)-\alpha}\left(1-e_{\alpha\ominus\mathrm{Re}_{\mu_*(s)}(z)}(t,s)\right)$$

so now let $t\to\infty$ and the Decay Theorem yields

$$\int_s^\infty |f(\tau)e_{\ominus z}(\sigma(\tau),s)|\,\Delta\tau \leq \frac{K}{\mathrm{Re}_{\mu_*(s)}(z)-\alpha}$$

for all $z\in\mathbb{C}_{\mu_*(s)}(\alpha)$. QED

Theorem

Let $f \in C_{rd}([s,\infty)\cap\mathbb{T},\mathbb{C})$ be of exponential order α. Then the Laplace transform $\mathscr{L}\{f\}$ converges uniformly in $\mathbb{C}_{\mu_*(s)}(\beta)$ where $\beta > \alpha$.

Proof.

Let $t \in [s,\infty)\cap\mathbb{T}$ and $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$. Using an inequality from our previous proof and the corollary to the Decay Theorem, we see

$$\begin{aligned}
\int_t^\infty |f(\tau)e_{\ominus z}(\sigma(\tau),s)|\Delta\tau &\leq \int_s^\infty |f(\tau)e_{\ominus z}(\sigma(\tau),s)|\Delta\tau \\
&\leq \frac{K}{\mathrm{Re}_{\mu_*(s)}(z)-\alpha} e_{\alpha\ominus \mathrm{Re}_{\mu_*(s)}(z)}(t,s) \\
&\leq \frac{K}{\mathrm{Re}_{\mu_*(s)}(z)-\alpha} \exp\left\{-[\mathrm{Re}_{\mu_*(s)}(z)-\alpha]m_{\mathrm{Re}_{\mu_*}}\right.
\end{aligned}$$

Proof.

If $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$ then $\mathrm{Re}_{\mu_*(s)}(z) > \alpha$ and thus for all $\tau \in [s,t) \cap \mathbb{T}$,

$$\frac{1}{1+\mu(\tau)\mathrm{Re}_{\mu*(s)}(z)} < \frac{1}{1+\mu(\tau)\alpha}.$$

Therefore

$$\begin{aligned} m_{\mathrm{Re}_{\mu_*(s)}(z)}(t,s) &= \int_s^t \frac{1}{1+\mu(\tau)\mathrm{Re}_{\mu_*(s)}(z)}\Delta\tau \\ &\leq \int_s^t \frac{1}{1+\mu(\tau)\alpha}\Delta\tau \\ &= m_\alpha(t,s). \end{aligned}$$

If $z \in \mathbb{C}_{\mu_*(s)}(\beta)$ for some $\beta > \alpha$ then $\mathrm{Re}_{\mu_*(s)}(z) > \beta$ and so $\alpha - \mathrm{Re}_{\mu_*(s)}(z) < \alpha - \beta$. Thus

$$\frac{1}{\mathrm{Re}_{\mu_*(s)}(z) - \alpha} \leq \frac{1}{\beta - \alpha} \text{ and } \alpha - \mathrm{Re}_{\mu_*(s)}(z) \leq \alpha - \beta.$$

Proof.

Combining these inequalities yields

$$\int_t^\infty |f(\tau)e_{\ominus z}(\sigma(\tau,s)|\Delta\tau \leq \frac{K}{\beta-\alpha}\exp\left\{-(\beta-\alpha)m_\alpha(t,s)\right\}.$$

Since we know that $\lim_{t\to\infty} m_\alpha(t,s)=\infty$, we see that a $t\in[s,\infty)\cap\mathbb{T}$ can be chosen to make the right-hand-side arbitrarily small. Therefore we have shown that for every $\epsilon>0$ there exists $r\in[s,\infty)\cap\mathbb{T}$ such that

$$\left|\int_t^\infty f(\tau)e_{\ominus z}(\sigma(t),s)\Delta\tau\right| \leq \epsilon$$

for all $t\in[r,\infty)\cap\mathbb{T}$ and all $z\in\mathbb{C}_{\mu_*(s)}(\beta)$. QED

Some further properties:

- (Convolution) Define $\hat{f}$ to be the solution of the shifting problem: for $t, t_0, s \in \mathbb{T}$ with $t \geq s \geq t_0$

$$u^{\Delta_t}(t, \sigma(s)) = -u^{\Delta_s}(t, s); u(t, t_0) = f(t)$$

  Define convolution by

$$(f * g)(t) = \int_{t_0}^{t} \hat{f}(t, \sigma(s)) g(s) \Delta s$$

  then

$$\mathscr{L}\{f * g\}(z) = \mathscr{L}\{f\}(z) \mathscr{L}\{g\}(z)$$

- (Inverse transform) Suppose $0 < \mu_{\min} \leq \mu(t) \leq \mu_{\max} < \infty$. If $\int_{c-i\infty}^{c+i\infty} |\mathscr{L}\{f\}(z)| \, |dz| < \infty$ and the poles of $\mathscr{L}\{f\}(z)$ are regressive constants $\{z_1, \ldots, z_n\}$ of finite order then

$$f(t) = \sum_{i=1}^{n} Res_{z=z_i} e_z(t, 0) \mathscr{L}\{f\}(z)$$

- (Transform of derivative)

$$\mathscr{L}\{y^{\Delta^n}\}(z) = z^n \mathscr{L}\{y\}(z) - \sum_{j=0}^{n-1} z^j y^{\Delta^{n-j-1}}(s)$$

- (Differentiation of transform) For $f \in C_{rd}([s,\infty) \cap \mathbb{T}, \mathbb{C})$,

$$-\frac{d}{dz}\mathscr{L}\{f\}(z;s) = \mathscr{L}\{m_z(\sigma(\cdot),s)f\}(z;s)$$

- (Integration) Define $v(x;t,s) := \int_x^\infty e_{x\ominus y}(t,s)dy$. If $\int_s^\infty e_\alpha(\tau,s)v(x;\sigma(t),s)e_{\ominus x}(\sigma(\tau),s)\Delta\tau < \infty$ for each $x \in \mathbb{R}_{\mu_*(s)}(\alpha)$ then for all $x \in \mathbb{R}_{\mu_*(s)}(\alpha)$

$$\int_x^\infty \mathscr{L}\{f\}(y;s)dy = \mathscr{L}\{v(x;\sigma(\cdot),s)f\}(x;s)$$

Laplace transform table

$y(t)$	$\mathscr{L}\{y\}(z)$
1	$\dfrac{1}{z}$
t	$\dfrac{1}{z^2}$
$h_k(t,s)$	$\dfrac{1}{z^{k+1}}$
$e_\alpha(t,s)$	$\dfrac{1}{z-\alpha}$

References

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