GPF Sequence Graphs

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Outline

1. Greatest Prime Factor
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   - Greatest Prime Factor Sequences
   - Result on GPF Sequences
2. GPF Sequence Graphs
   - Graph Theory
   - GPF Sequence Graphs
   - Results
3. Conjectures and Empirical Evidence
   - Counting Connected Components

Fundamental Theorem of Arithmetic

- A natural number n is an element of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- A prime number $p \in \mathbb{P}$ is a natural number such that its factors are 1 and itself.
- Any natural number, n, has a unique prime factorization $n = p_1^{e_1} \ldots p_m^{e_m}$ for some natural number m and prime numbers $p_1 \ldots p_m$.

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Greatest Prime Factor Function

- $gpf(n) : \mathbb{N} - \{1\} \rightarrow \mathbb{P}$
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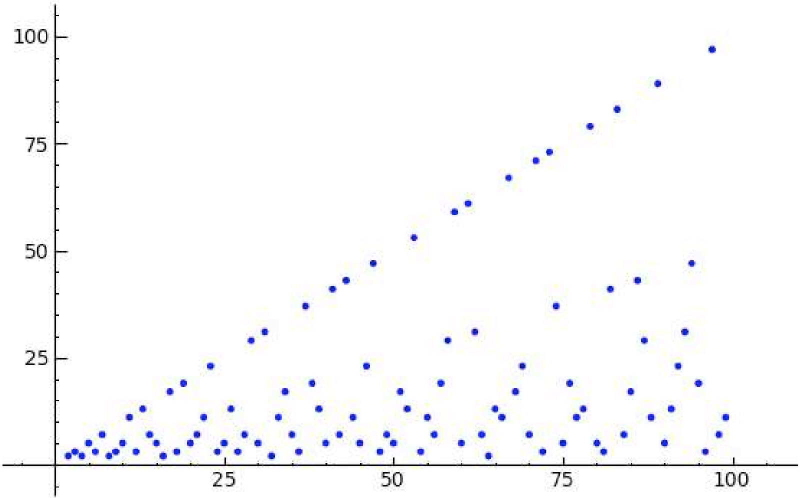

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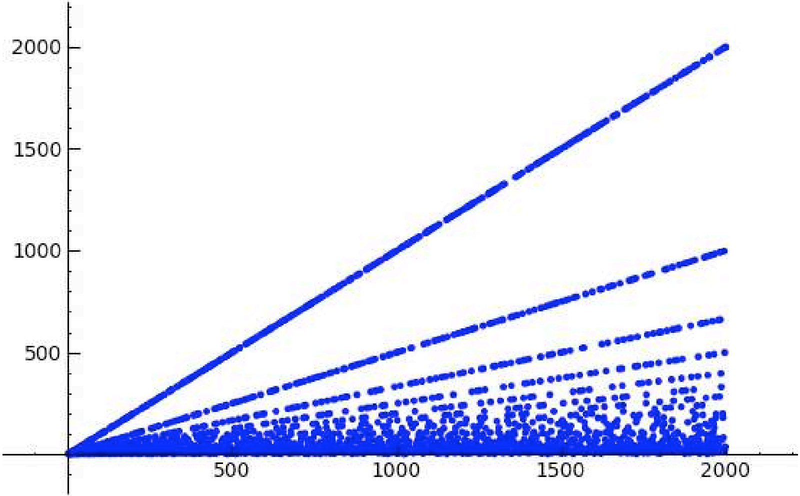

Greatest Prime Factor Sequence

- A greatest prime factor sequence is defined as $x_{n+1} = gpf(ax_n + b)$ where x_0 is prime, and $a, b \in \mathbb{N}$.
- For example, let $x_0 = 2$, $a = 3$, $b = 1$. Then $x_n = 2, 7, 11, 17, 13, 5, 2, 7, 11, 17, 13, 5, 2, \ldots$.
- Let $x_0 = 7$, $a = 5$, $b = 3$. Then $x_n = 7, 19, 7, 19, 7, 19, 7, 19, 7, 19, \ldots$.
- Let $x_0 = 9$, $a = 2$, $b = 1$. Then $x_n = 9, 19, 13, 3, 7, 5, 11, 23, 47, 19, 13, 3, \ldots$.

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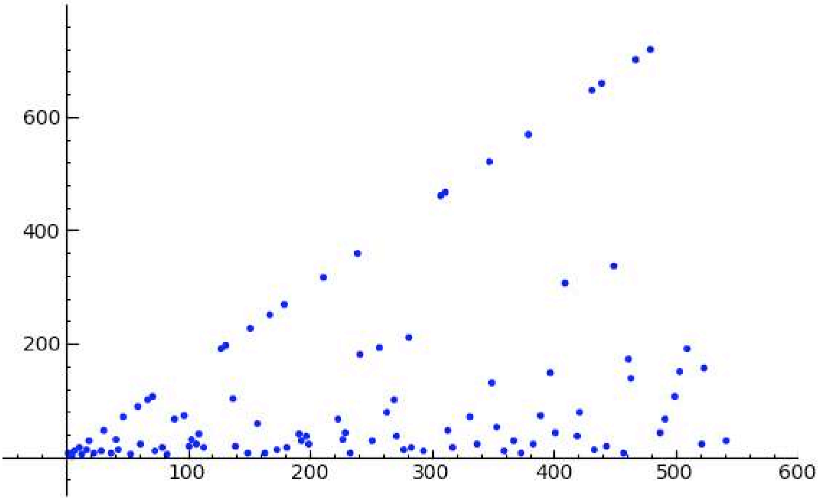
Greatest Prime Factor Function

$gpf(3a+1)$ for prime a



A Conjecture and Result on GPF Sequences

- Caragiu and Sheckelhoff conjectured that all gpf sequences are ultimately periodic.
- They showed it to be true for a particular family of functions.

Theorem

If $a = 1$ and $b \in \mathbb{N}$, then if x_0 is prime, $x_{n+1} = gpf(x_n + b)$ is ultimately periodic

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Some graph theory terms

A *graph* is a set, $G = \{V(G), E(G)\}$, where $V(G)$ is a set of *vertices* and $E(G)$ is a set of *edges* between those vertices.

A *path* in a graph is a sequence of edges, $a_0, a_1, \ldots, a_n$ such that (a_i, a_{i+1}) is a valid edge.

A *cycle* in a graph is a path, $a_0, \ldots, a_n$ such that $a_0 = a_n$.

GPF Sequence Graphs

- We want to graphically represent the sequences.
- Define $Uax + b$ where $a, b \in \mathbb{N}$, is the graph obtained by taking graph unions between gpf sequences of all possible initial sequences where the nodes are the prime numbers, and edges are adjacent sequence members.
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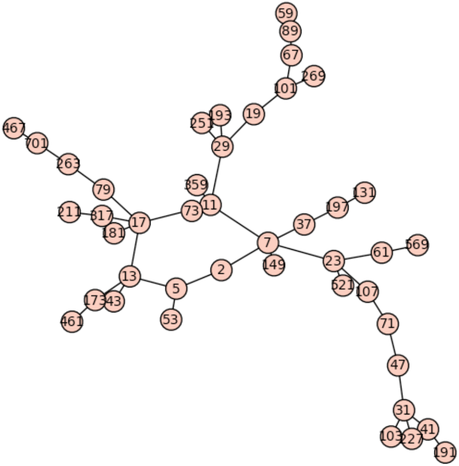
Some Examples

U_{3x+1}



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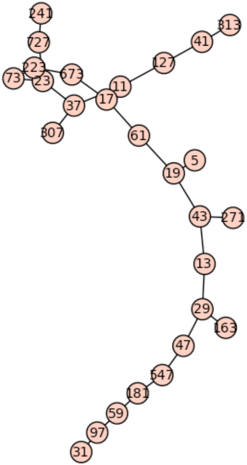
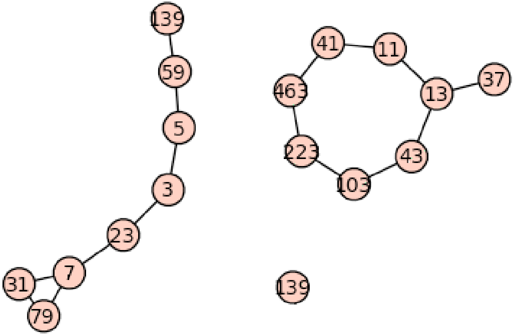
U_{3x+4}



Some Examples

U_{2x+17}



Results

Theorem

For $U\,ax + b$ and for any $m, n \in \mathbb{N}$, any n-cycle A and any m-cycle B, $A \bigcap B \neq \emptyset$ if and only if $A = B$.

Proof.

$(\rightarrow)$ Suppose that there exists an *m*-cycle, A and an *n*-cycle, B such that $A \bigcap B \neq \emptyset$. Then, there exists a node, $t \in A \bigcap B$. Since $x \in A$, we know that $gpf(at + b) \in A$ and since $x \in B$, we know that $gpf(at + b) \in B$. Thus, $gpf(at + b)$ is also in the intersection. It follows inductively that $A = B$ if $A \bigcap B \neq \emptyset$. □

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Theorem

For U_{ax+b}, there does not exist a path between any disjoint cycles A and B.

Proof.

Suppose an n-node path, $a_1, \ldots, a_n$ exists from A to B such that $a_0 \in A$ and other $a_i \notin A$ and $a_n \in B$ with other $a_i \notin B$. Since a_0 attached to a_1, $gpf(a \cdot a_1 + b) = a_0$. Then, we must have $gpf(a \cdot a_2 + b) = a_1$, and likewise $gpf(a \cdot a_i + b) = a_{i-1}$ for $i \leq 1 < n$. Consider $gpf(a \cdot a_n + b)$. To preserve the path, $gpf(a \cdot a_n + b) = a_{n-1}$, but $a_{n-1} \notin B$. Contradiction! □

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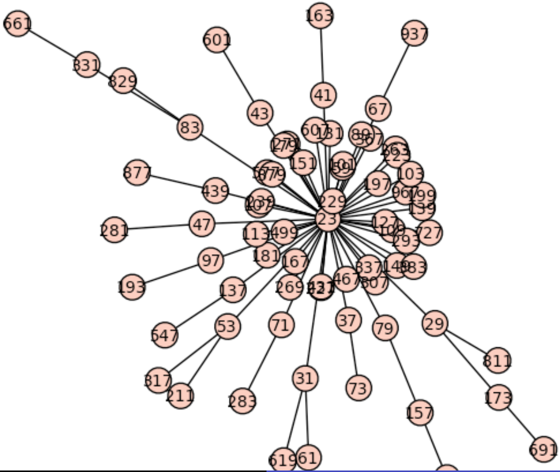
Number of Connected Components?

Conjecture: U_{ax+a} has only one connected component.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
1	1	1	1	1	1	2	1	1	2	1	1	1	2	3	1	2	1	3	1	1	2
2	2	1	3	1	1	1	4	1	1	1	1	2	3	1	3	1	3	2	1	1	4
3	2	3	1	1	4	1	3	3	1	3	2	1	1	3	1	2	4	2	2	3	1
4	1	2	1	1	1	3	1	1	3	1	1	1	2	4	2	1	1	1	3	1	1
5	2	4	3	4	1	4	4	2	2	1	3	2	1	3	1	1	2	1	3	1	1
6	3	2	1	3	3	1	5	1	2	4	1	1	3	3	1	3	1	1	2	3	3
7	2	3	3	3	2	2	1	2	4	1	2	2	3	1	5	1	4	3	3	2	1
8	2	1	2	2	4	1	3	1	1	1	2	3	3	1	1	1	5	3	2	1	1
9	2	2	1	3	2	3	2	3	1	4	3	1	3	2	4	5	1	1	2	1	3
10	2	2	3	4	1	3	2	4	2	1	3	4	4	4	2	2	2	2	2	1	3
11	2	2	2	5	2	3	2	4	3	3	1	1	4	5	5	2	5	2	2	3	4
12	2	3	1	2	2	1	1	3	1	3	1	1	3	5	1	1	3	2	1	4	1
13	2	2	2	3	3	3	2	2	2	4	3	2	1	4	2	1	2	3	3	5	4
14	3	2	3	3	3	3	1	3	3	2	2	2	3	1	3	2	3	4	5	1	2
15	4	3	2	2	1	4	4	1	3	2	3	4	6	2	1	4	2	4	4	1	4
16	1	2	1	1	2	2	2	2	3	4	1	1	1	3	4	1	1	1	2	1	3
17	2	3	2	2	1	1	2	2	3	2	1	2	2	3	3	3	1	3	4	3	1
18	1	2	2	2	3	1	4	3	1	2	1	3	1	2	2	3	4	1	1	4	5
19	3	2	3	2	4	1	2	2	3	4	3	2	3	4	3	3	3	3	1	2	3
20	3	2	2	2	1	3	2	4	1	1	4	3	1	2	1	4	3	2	3	1	3
21	4	3	2	2	3	3	1	2	3	3	5	3	2	2	2	3	5	2	2	4	1

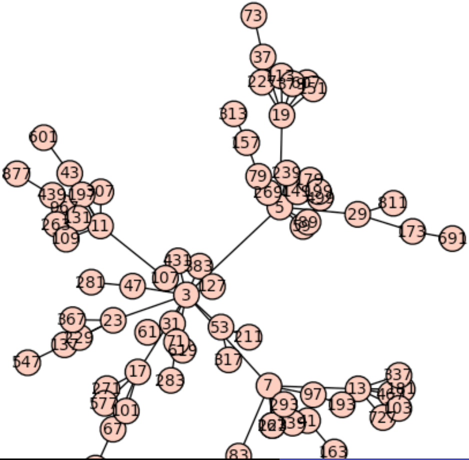
Connected Components

U_{23x+23}



Connected Components

U_{6x+6}



Thank you!

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