What is a Meijer-G function?

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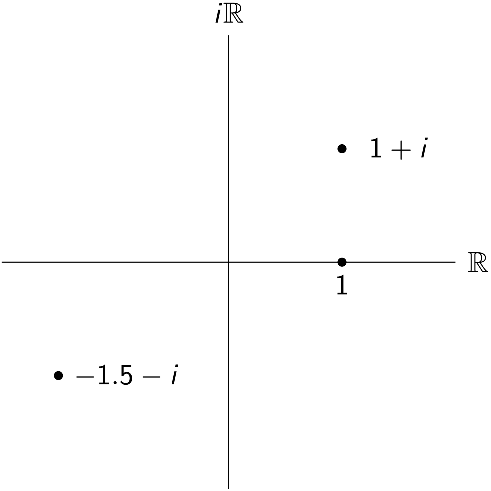
Complex numbers $\mathbb{C}$: $i = \sqrt{-1}$



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A **residue at a simple pole** z_0 **of a function** f is $\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)$.

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Consider $f(z) = \dfrac{1}{z^2+1}$. Algebra $\rightarrow$ $z^2 + 1 = 0$ has solutions $z = \pm i$. So f has poles at $z = i$ and $z = -i$. Factor: $z^2 + 1 = (z+i)(z-i)$.

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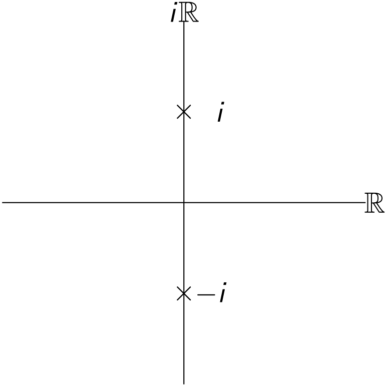

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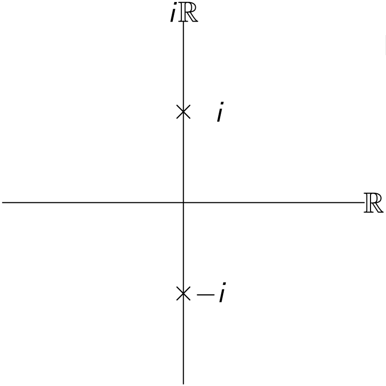

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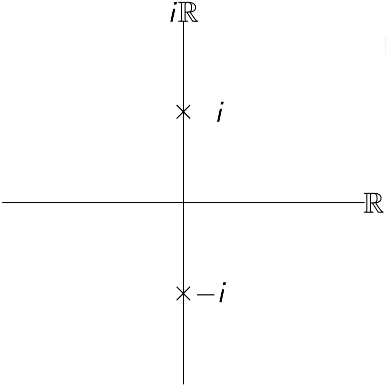

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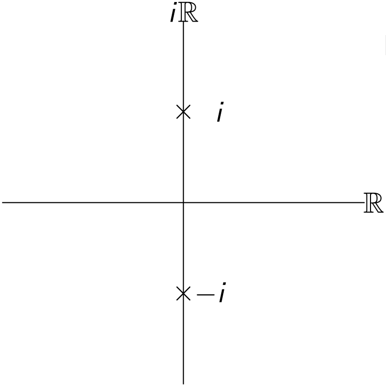

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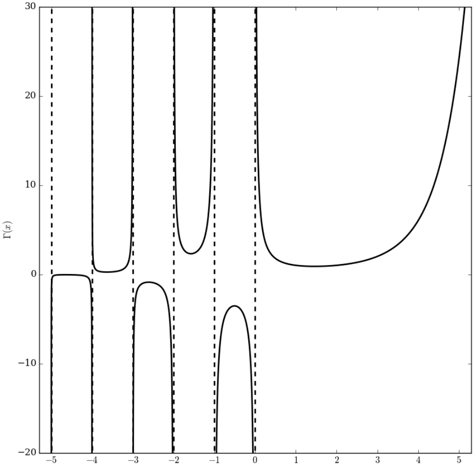
Gamma function

Gamma function: $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x, \quad \mathrm{Re}(z) > 0$



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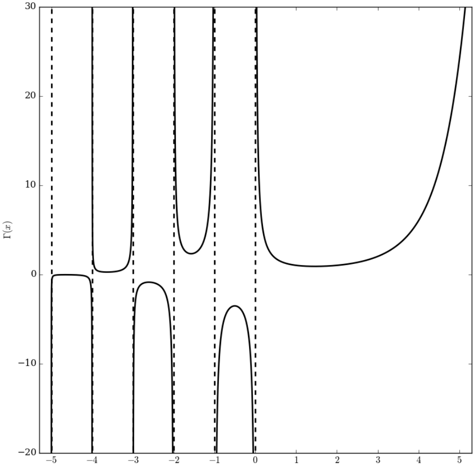

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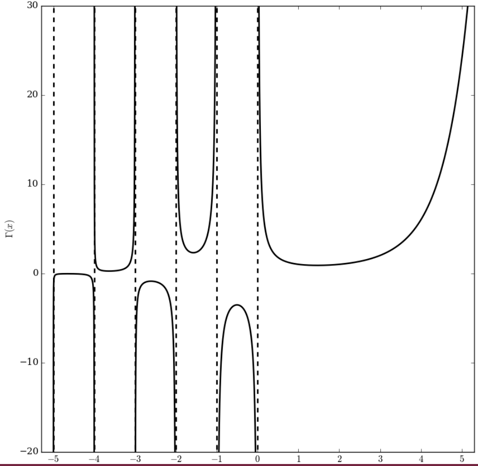
Factorials: $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n!$



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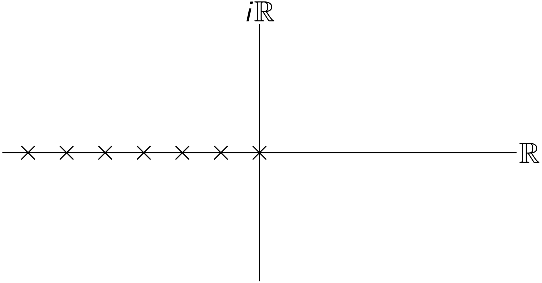
Poles: $z = 0, -1, -2, \ldots$



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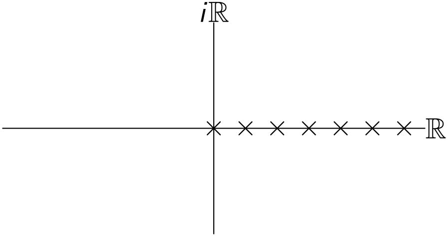

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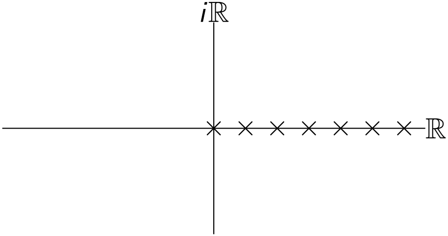

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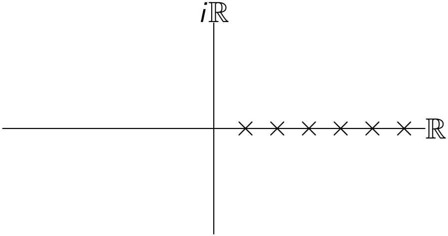

Where's $\Gamma(1 - z)$ have poles?

Whenever $1 - z = -n$, i.e. at $z = 1 + n$ for $n = 0, 1, 2, \ldots$

Residues (recall $z\Gamma(z) = \Gamma(z+1)$)

Example (residue of Γ at -1): Note $(z+1)\Gamma(z+1) = \Gamma(z+2)$ so

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\frac{\Gamma(z+2)}{z}}{z+1} =: \frac{\Phi(z)}{z-(-1)}$$

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$$\operatorname*{Res}_{z=-1} \Gamma(z) = \Phi(-1) = \frac{\Gamma(-1+2)}{-1} = \frac{\Gamma(1)}{-1} = -1.$$

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Similar formulas:

$$\operatorname*{Res}_{z=n}\Gamma(1-z) = \frac{(-1)^n}{(n-1)!} \text{ and } \operatorname*{Res}_{z=n}\Gamma(-z) = \frac{(-1)^{n+1}}{n!}$$

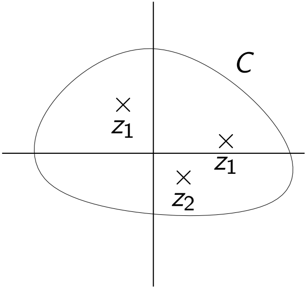
Residue theorem

Surprisingly, residues can evaluate what are called "contour integrals".

Theorem (Residue theorem)

If C is a simple closed contour oriented positively and f is analytic in C except for a finite number of poles z_1, ..., z_k inside C, then*

$$\int_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

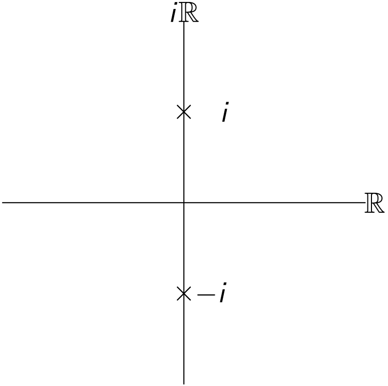


*** we will apply it to infinite sets to simplify analysis!!**

Computing integrals with the residue theorem

Recall $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)}$: earlier we had

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2i} \text{ and } \operatorname*{Res}_{z=i} f(z) = -\frac{1}{2i}$$



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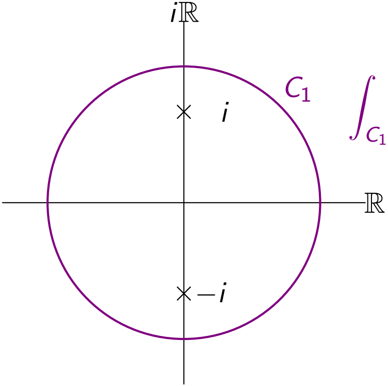

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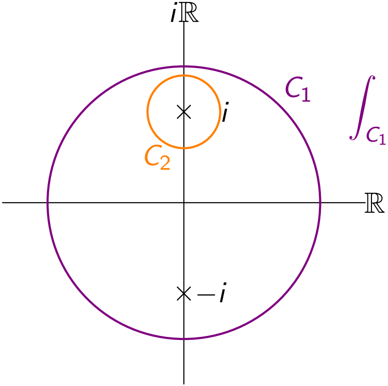

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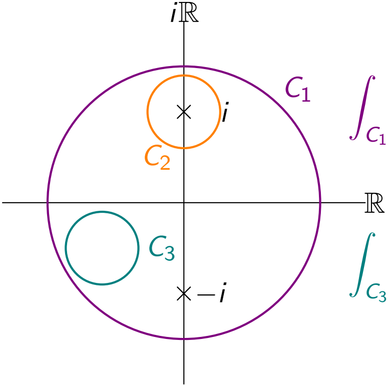

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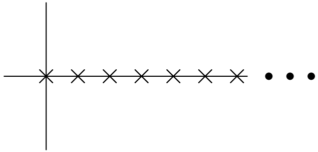
$$\int_{C_3} f(z)\mathrm{d}z = 2\pi i \left[\text{no residues!}\right] = 0$$

Contour integrals defining functions

Consider $\int_C \Gamma(-s)x^s \mathrm{d}s$. Only poles here come from $\Gamma(-s)$. Recall

$$\operatorname*{Res}_{s=n}\Gamma(-s) = \frac{(-1)^{n+1}}{n!}.$$



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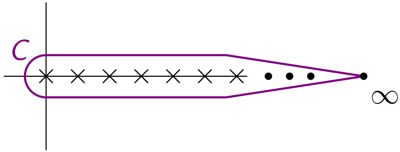

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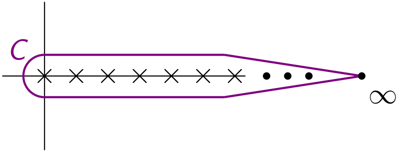

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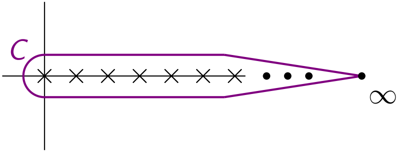

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Hence:

$$e^x = -\frac{1}{2\pi i}\int_C \Gamma(-s)(-x)^s \mathrm{d}s$$

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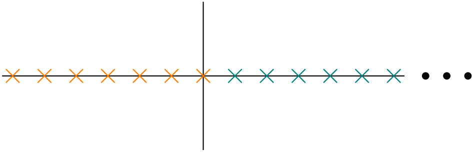
We get poles from both $\Gamma(1-s)$ (s=1,2,3,...) and $\Gamma(s)$ (s=0,-1,-2,...).



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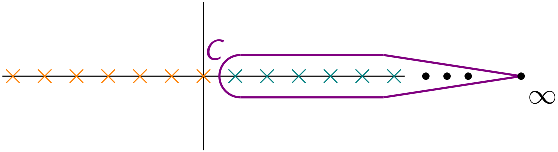
Take C to surround only the poles of $\Gamma(1-s)$.



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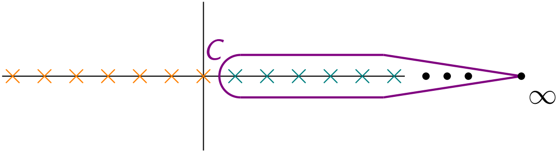

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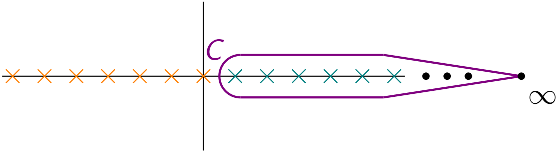

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Hence:

$$\ln(x+1) = -\frac{1}{2\pi i} \int_C \frac{\Gamma(1-s)\Gamma(s)^2}{\Gamma(1+s)} x^s \mathrm{d}s$$

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2. What argument should be given to the Γ functions?
3. Which poles should be integrated?

These items are specified by the Meijer-G function!

$$G^{m,n}_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$$

$$=$$

$$-\frac{1}{2\pi i}\int_C \frac{\left[\overbrace{\Gamma(b_1 - s)\ldots\Gamma(b_m - s)}^{\text{only these ones*}}\right]\left[\Gamma(1-a_1+s)\ldots\Gamma(1-a_n+s)\right]}{\left[\Gamma(1-b_{m+1}+s)\ldots\Gamma(1-b_q+s)\right]\left[\Gamma(a_{n+1}-s)\ldots\Gamma(a_p-s)\right]} x^s \mathrm{d}s$$

*** note: there's more choices; this choice sometimes fails!**

A wonderful class of functions!

Meijer G representations exist for many functions

1. $\cos(x) = \sqrt{\pi} G^{1,0}_{0,2}(0; ; -x)$
2. $\arctan(x) = \frac{1}{2} G^{1,2}_{2,2}\left(\frac{1}{2}, 1; \frac{1}{2}, 0; x^2\right)$
3. (Bessel functions) $J_\nu(x) = G^{1,0}_{0,2}\left(; \frac{\nu}{2}, -\frac{\nu}{2}; \frac{x^2}{4}\right)$

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2. $\arctan(x) = \frac{1}{2} G^{1,2}_{2,2}\left(\frac{1}{2}, 1; \frac{1}{2}, 0; x^2\right)$
3. (Bessel functions) $J_\nu(x) = G^{1,0}_{0,2}\left(; \frac{\nu}{2}, -\frac{\nu}{2}; \frac{x^2}{4}\right)$

Strong closure properties: if $h(x)$ and $\ell(x)$ are Meijer-G functions, then…

1. $h(-x)$, $h(\frac{1}{x})$, $x^a h(x)$, and $h'(x)$ are all Meijer-G functions
2. $\int_c^x h(t)\mathrm{d}t$ is a Meijer-G function (for some c)
3. the convolution $(h * \ell)(x)$ is a Meijer-G function

Thank you

Thank you for attending.

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