Dynamic Matrix Exponential via Matrix Cylinder Transformation

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Preliminaries

If $A \in \mathbb{C}^{n \times n}$ is diagonalizable, then Jordan canoncial form is

$$A = J^{-1}\mathrm{diag}(\lambda_1, \ldots, \lambda_n)J,$$

where the λ_i are eigenvalues of A. If $f\colon U \subset \mathbb{C} \to \mathbb{C}$ is a function so that $f(\lambda_i)$ exists for $i = 1, 2, \ldots, n$, then we define the matrix analogue of f by

$$f(A) = J^{-1}\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))J.$$

note: the formula exists but is more complicated for nondiagonalizable matrices

We easily obtain $\mathrm{Log}(A)$, $\exp(A)$, and $|A|$.
We also extend the function $\mathrm{sign}\colon \mathbb{C} \setminus \{z \in \mathbb{C}\colon \mathrm{Re}(z) = 0\} \to \{1, -1\}$ defined by

$$\mathrm{sign}(z) = \begin{cases} 1, & \mathrm{Re}(z) > 0 \\ -1, & \mathrm{Re}(z) < 1. \end{cases}$$

To a matrix sign function.

Preliminaries

Theorem ([5],Theorem 10.2)

For $A, B \in \mathbb{C}^{n\times n}$,

$$\exp((A+B)t) = \exp(At)\exp(Bt)$$

for all $t \in \mathbb{C}$ *iff* $AB = BA$.

Theorem ([5],Theorem 11.3)

Suppose $B, C \in \mathbb{C}^{n\times n}$ *have no negative eigenvalues and* $BC = CB$. *If for every eigenvalue* λ_j *of* B *and corresponding eigenvalue* η_j *of* $\mathbb{C}$,

$$\arg(\lambda_j) + \arg(\eta_j)| < \pi,$$

then $\mathrm{Log}(BC) = \mathrm{Log}(B) + \mathrm{Log}(C)$.

Preliminaries

Using standard branch cut $(-\infty, 0]$, we obtain logarithms at $x < 0$ as

$$\log(x) = \log(|x|) + i\pi.$$

From this, if A is Hermitian (hence diagonalizable), then

$$\mathrm{Log}(A) = J^{-1}\Bigg(\mathrm{diag}(\mathrm{Log}(|\lambda_1|), \ldots, \mathrm{Log}(|\lambda_n|)$$
$$+\mathrm{diag}\left(\frac{1-\mathrm{sign}(\lambda_1)}{2}, \ldots, \frac{1-\mathrm{sign}(\lambda_n)}{2}\right) i\pi\Bigg) J,$$

or more compactly,

$$\mathrm{Log}(A) = \mathrm{Log}(|A|) + i\pi\frac{I - \mathrm{sign}(A)}{2}.$$

Preliminaries

Let $\mathbb{T}$ be a time scale.

A function $f: \mathbb{T} \to \mathbb{C}$ is called regressive if $\forall t \in \mathbb{T}$, $1 + \mu(t)f(t) \neq 0$.

More generally, a function $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ is called regressive if $\forall t \in \mathbb{T}^{\kappa}$, $1 + \mu(t)A(t)$ is invertible.

Sometimes we will use the symbol a_ℓ to denote right-scattered points.

Scalar cylinder transformation ξ

Hilger complex plane ("regressive constants" with respect to h) is

$$\mathbb{C}_h = \begin{cases} \mathbb{C} \setminus \left\{ -\frac{1}{h} \right\}, & h > 0, \\ \mathbb{C}, & h = 0. \end{cases}$$

and the strip

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} \colon -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\}.$$

Cylinder transformation $\xi_h \colon \mathbb{C}_h \to \mathbb{Z}_h$ is

$$\xi_h(z) = \begin{cases} \frac{1}{h} \log(1 + hz), & h > 0, \\ z, & h = 0. \end{cases}$$

Preliminaries

Let $\mathcal{F} = \{[a, b) \cap \mathbb{T} \colon a, b \in \mathbb{R}\}$ (ring of subsets of $\mathbb{T}$) and define $m_1([a, b)) = b - a$.

The unique Carathéodory extension of m_1 to the σ-algebra generated by $\mathcal{F}$ is called the μ_Δ measure.

This measure expresses the Δ-integral as a "Lebesgue Δ-integral" allowing us to define the function space $L^1_{\mathrm{loc}}(\mathbb{T}, \mu_\Delta)$, the locally Δ-integrable functions.

Scalar dynamic exponential e_p

For certain $p\colon \mathbb{T} \to \mathbb{C}$, exponential function $e_p\colon \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right).$$

Coincidentally, if $y(t) = e_p(t,s)$, then y satisfies the initial value problem

$$y^\Delta = py, \quad y(s) = 1.$$

Typically p is chosen to be regressive and rd-continuous, but in [4], the exponential was studied for $p \in L^1_{\mathrm{loc}}(\mathbb{T}, \mu_\Delta)$.

Matrix dynamic exponential e_A

For certain $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$, $e_A\colon \mathbb{T} \times \mathbb{T} \to \mathbb{C}^{n\times n}$ is **defined** to be the solution of the matrix dynamic equation

$$y^{\Delta} = Ay, \quad y(s) = I.$$

No matrix cylinder transformation is typically mentioned.

Matrix cylinder transformation Ξ

$\mathbb{C}_h^{n\times n}$: set of matrices X in $\mathbb{C}^{n\times n}$ for which $1 + hX$ is invertible
$\mathbb{Z}_h^{n\times n}$: set of $\mathbb{C}^{n\times n}$ matrices whose eigenvalues lie in $\mathbb{Z}_h$
Matrix cylinder transformation Ξ: $\mathbb{C}_h^{n\times n} \to \mathbb{Z}_h^{n\times n}$ is

$$\Xi_h(A) = \begin{cases} \frac{1}{h}\mathrm{Log}\left(1 + hA\right), & h > 0, \\ A, & h = 0. \end{cases}$$

Immediate Consequence: Ξ applies the scalar cylinder transformation ξ to the eigenvalues of A

Matrix cylinder transformation

Lemma

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ is a regressive Hermitian matrix function, then

1. $I + \mu(t)A(t)$ *is Hermitian for all* $t \in \mathbb{T}$,
2. *if* $A(t)$ *is positive semi-definite for all* $t \in \mathbb{T}$, *then* $\Xi_{\mu(t)}(A(t))$ *is Hermitian for all* $t \in \mathbb{T}$, *and*
3. *if* $A(t)$ *is positive definite for all* $t \in \mathbb{T}$, *then* $\Xi_{\mu(t)}(A(t))$ *is positive definite for all* $t \in \mathbb{T}$.

Proof is routine calculation.

Lemma

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ *is a regressive Hermitian matrix function with each* $a_{ij} \in L^1_{\mathrm{loc}}(\mathbb{T}, \mu_\Delta)$, *then the function* $t \mapsto \Xi_{\mu(\tau)}(A(t))$ *is locally integrable.*

Proof.

Since A is locally integrable $\displaystyle\sum_{t_0 \le a_\ell < t} \mu(a_\ell)\|A(a_\ell)\|_1 \le \int_{t_0}^{t} \|A(\tau)\|_1 \Delta\tau < \infty.$

So $\mu(a_\ell)\|A(a_\ell)\|_1 \le \dfrac{1}{2}$ for all but finitely many indices ℓ. If $\mu(\tau) = 0$, then $\|\Xi_{\mu(\tau)}(A(\tau))\|_1 = \|A(\tau)\|_1$. Now suppose $\mu(\tau) > 0$. Then $\tau = a_\ell$ for some ℓ. By definition, $\left\|\Xi_{\mu(\tau)}(A(\tau))\right\|_1 = \dfrac{1}{\mu(a_\ell)} \left\|\mathrm{Log}(I + \mu(a_\ell)A(a_\ell))\right\|_1$. We know

$$|\ln(1+y)| \le \begin{cases} y, & 0 \le y \le \frac{1}{2} \\ |y| \ln(4), & -\frac{1}{2} \le y < 0. \end{cases}$$

Matrix cylinder transformation

Proof.

Since $r\Big(\mu(a_\ell)A(a_\ell)\Big) \leq \|\mu(a_\ell)A(a_\ell)\|_1 \leq \frac{1}{2}$, we use power seriees for matrix logarithm to write

$$\begin{aligned}\|\mathrm{Log}(I+\mu(a_\ell)A(a_\ell)\|_1 &= \left\|\sum_{k=1}^{\infty}\frac{(-1)^k(\mu(a_\ell)A(a_\ell))^k}{k}\right\| \\ &\leq \sum_{k=1}^{\infty}\frac{(\mu(a_\ell)\|A(a_\ell)\|_1)^k}{k} \\ &= -\ln\left(1-\mu(a_\ell)\|A(a_\ell)\|_1\right) \\ &\leq \ln(4)\mu(a_\ell)\|A(a_\ell)\|_1.\end{aligned}$$

Therefore

$$\|\Xi_{\mu(a_\ell)}(A(a_\ell))\|_1 = \frac{1}{\mu(a_\ell)}\|\mathrm{Log}(I+\mu(a_\ell)A(a_\ell))\|_1 \leq \ln(4)\|A(a_\ell)\|_1.$$

Matrix cylinder transformation

Lemma

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ *is a regressive Hermitian matrix function with each* $a_{ij} \in L^1_{\mathrm{loc}}(\mathbb{T}, \mu_\Delta)$, *then the function* $t \mapsto \Xi_{\mu(\tau)}(A(t))$ *is locally integrable.*

Proof.

Hence,

$$\begin{aligned}\left\| \int_{t_0}^{t} \Xi_{\mu(\tau)}(A(\tau))\Delta\tau \right\|_1 &\le \int_{t_0}^{t} \left\|\Xi_{\mu(\tau)}(A(\tau))\right\|_1 \Delta\tau \\ &\le \ln(4) \int_{t_0}^{t} \|A(\tau)\|_1 \Delta\tau < \infty,\end{aligned}$$

completing the proof. □

New matrix exponential E_A

For $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ with locally integrable components, we define $E_A\colon \mathbb{T}\times\mathbb{T} \to \mathbb{C}^{n\times n}$ by

$$E_A(t,s) = \exp\left(\int_s^t \Xi_{\mu(\tau)}(A(\tau))\Delta\tau\right).$$

Proposition

If A is regressive, then $E_A(t,t) = I$ and $E_{\mathbf{0}}(t,s) = I$.

Proposition

If A is regressive, then $E_A^{-1}(t,s) = E_A(s,t)$.

Example

In $\mathbb{T} = \mathbb{N}_0$ taking $A(t) = \begin{bmatrix} t+2 & (t+2)^2 \\ (t+2)^3 & (t+2)^4 \end{bmatrix}$, we obtain

$$e_A(1,0) = I + A(0) = \begin{bmatrix} 3 & 4 \\ 8 & 17 \end{bmatrix} = E_A(1,0),$$

and

$$e_A(2,0) = (I + A(1))(I + A(0)) = \begin{bmatrix} 84 & 169 \\ 737 & 1502 \end{bmatrix},$$

while

$$\begin{aligned} E_A(2,0) &= \exp\left(\mathrm{Log}(I + A(0)) + \mathrm{Log}(I + A(1))\right) \\ &= \exp\left(\begin{bmatrix} \frac{\ln(19)}{9} + \frac{\ln(85)}{28} & \frac{2\ln(19)}{9} + \frac{3\ln(85)}{28} \\ \frac{4\ln(19)}{9} + \frac{9\ln(85)}{28} & \frac{8\ln(19)}{9} + \frac{27\ln(85)}{28} \end{bmatrix}\right) \\ &\approx \begin{bmatrix} 96.9684 & 240.686 \\ 582.728 & 1463.05 \end{bmatrix} \end{aligned}$$

Properties

Properties of e_A and E_A

$e_A(t, t_0)$	$E_A(t, t_0)$
$e_A(t, t) = I$	$E_A(t, t) = I$
$e_0(t, t_0) = I$	$E_0(t, t_0) = I$
$e_A(t, t_0) = e_A(t_0, t)^{-1}$	$E_A(t, t_0) = E_A(t_0, t)^{-1}$
$e_A^\sigma = (I + \mu A)e_A$	$E_A^\sigma = \exp\left(\mu \mathrm{Log}(I + \mu A) + \mathrm{Log}(E_A)\right)$

Continuous and discrete part

Theorem

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ *is regressive, Hermitian matrix function with locally integrable components, then for all* $t \in \mathbb{T}$ *with* $t \geq t_0$*,*

$$E_A(t, t_0) = \exp\left(\int_{[t_0,t]\cap\mathbb{T}} A(\tau)\mathrm{d}\tau + \sum_{t_0 \leq a_\ell < t}\left[\mathrm{Log}(|I + \mu(a_\ell)A(a_\ell)|) + i\pi\frac{I - \mathrm{sign}(I + \mu(a_\ell)A(a_\ell))}{2}\right]\right)$$

The scalar case then has, for m dependent on t,

Corollary

$$e_p(t, t_0) = (-1)^m \exp\left(\int_{[t_0,t]\cap\mathbb{T}} p(\tau)\mathrm{d}\tau\right) \prod_{t_0 \leq a_\ell < t} (1 + \mu(a_\ell)p(a_\ell))$$

Trace Inequality

Theorem ([2], (8))

If A and B are positive semi-definite, then

$$\mathrm{Tr}\Big(\exp(A+B)\Big) \leq \mathrm{Tr}\Big(\exp(A)\exp(B))\Big).$$

Corollary

$$\mathrm{Tr}(E_A(t,t_0))$$
$$\leq \mathrm{Tr}\left(\exp\left(\int_{[t_0,t]\cap\mathbb{T}} A(\tau)\mathrm{d}\tau\right)\exp\left(\sum_{t_0\leq a_\ell<t}\mathrm{Log}(1+\mu(a_\ell)A(a_\ell))\right)\right)$$

Distributing exp through E_A

Theorem

Let $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ be a regressive Hermitian matrix function where a_{ij} are locally integrable. If A satisfies

1. *for all $t_1, t_2 \in [t_0, t) \cap \mathbb{T}$, $A(t_1)$ commutes with $A(t_2)$, and*
2. *for all $t_1 \in [t_0, t) \cap \mathbb{T}$, $\displaystyle\int_{[t_0,t)\cap\mathbb{T}} A(\tau)\mathrm{d}\tau$ commutes with $A(t_1)$,*

then

$$E_A(t, t_0) = \exp\left(\int_{[t_0,t]\cap\mathbb{T}} A(\tau)\mathrm{d}\tau\right) \prod_{t_0 \le a_\ell < t} (I + \mu(a_\ell)A(a_\ell)).$$

Measurable eigenvectors

Theorem

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ is a matrix function with each $a_{ij} \in L^1_{\mathrm{loc}}(\mathbb{T}, \mu_\Delta)$ and $A(t)$ is Hermitian for every $t \in \mathbb{T}$, then the distinct eigenvalues $\lambda_n(t) > \lambda_{n-1}(t) > \ldots > \lambda_{n-m}(t)$ of $A(t)$, where $m \leq n$, are measurable and have corresponding measurable eigenvector functions which form an orthonormal basis of $\mathbb{C}^{n\times n}$.

Diagonalizer

The previous theorem is needed to prove the existence of U in the following theorem.

Theorem

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ is a matrix function with each a_{ij} locally integrable and $A(t)$ is positive definite for all $t \in \mathbb{T}$, then there is a μ_Δ-measurable unitary matrix function $U(t)$ satisfying

$$E_{U*AU}(t, t_0) = \operatorname{diag}\left(e_{\lambda_1}(t, t_0), \ldots, e_{\lambda_n}(t, t_0)\right),$$

where the λ_i are the (not necessarily distinct) eigenvalues functions of A.

Dynamic integral equation

Earlier:

Lemma

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ *is a regressive Hermitian matrix function with each* $a_{ij} \in L^1_{\mathrm{loc}}(\mathbb{T}, \mu_\Delta)$*, then the function* $t \mapsto \Xi_{\mu(\tau)}(A(t))$ *is locally integrable.*

Thus the integral of Ξ is locally absolutely continuous allowing us to prove

Lemma

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ *is regressive Hermitian with each* $a_{ij} \in L^1_{\mathrm{loc}}(\mathbb{T}, \mu_\Delta)$*, then the function* $t \mapsto A(t)E_A(t, t_0)$ *is locally integrable.*

Dynamic integral equation

Theorem

If $A\colon \mathbb{T} \to \mathbb{C}^{n\times n}$ *is a regressive Hermitian matrix function with each* a_{ij} *locally integrable, then the function* $Y(t) = E_A(t, t_0)$ *satisfies*

$$Y(t) = I + \int_{t_0}^{t} F(\tau, Y(\tau))\Delta\tau,$$

where

$$F(t, Y(t)) = \begin{cases} A(t)Y(t), & \mu(t) = 0, \\ \dfrac{\exp\left(\operatorname{Log}(I + \mu(t)A(t)) + \operatorname{Log}(Y(t))\right) - Y(t)}{\mu(t)}, & \mu(t) > 0. \end{cases}$$

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Thanks

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