An introduction to nonstandard topology

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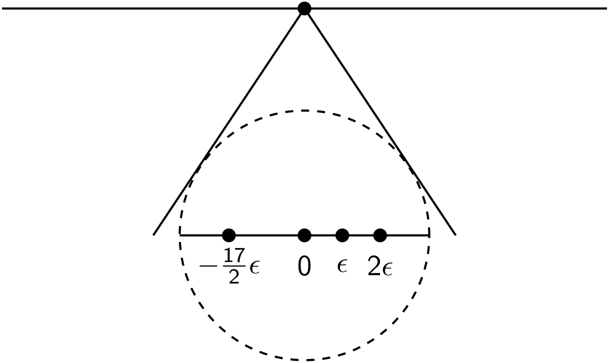
- Abraham Robinson (1966)

Nonstandard analysis

Important concept: monads! Below is the "monad of 0", written $\mu(0)$.



Formal construction by ultraproducts; e.g. can take $\epsilon = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$.

Nonstandard extensions

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Real numbers: $\mathbb{R}$ extends to ${}^*\mathbb{R}$, contains monads of all points, contains reciprocals of infinitesimals ("infinite numbers")

Transfer principle

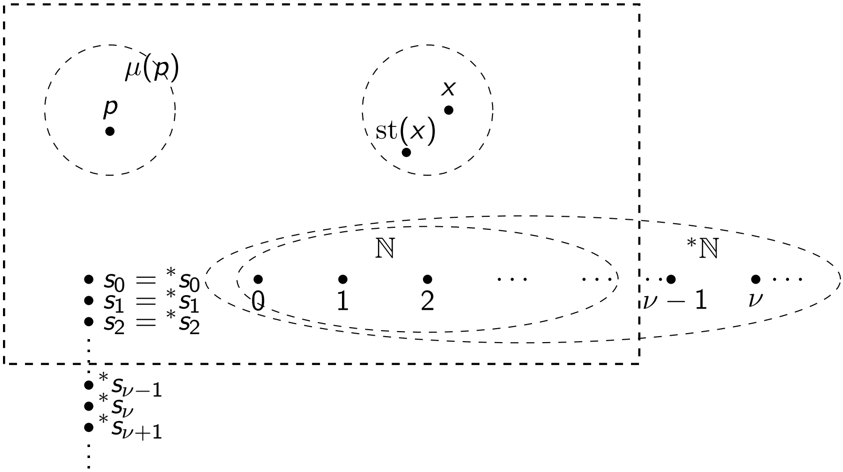
Standard convergence

$$a_n \to L \text{ means } \forall \epsilon > 0 \exists N \forall n \geq N |a_n - L| < \epsilon$$

Nonstandard convergence

$$a_n \to L \text{ means } \forall H \in {}^*\mathbb{N} \setminus \mathbb{N}, a_H \in \mu(L)$$

Various nonstandard notions in one picture

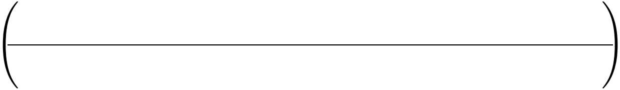


Topology

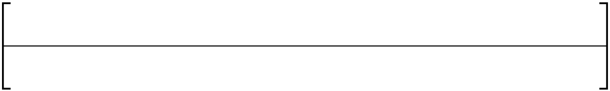
- If X is a set, a topology on X (called τ) is a collection of subsets of X containing $\emptyset$, containing X, closed under arbitrary unions, and closed under finite intersections.
- (X, τ) is called a **topological space**.
- A topological space (X, τ) is called **compact** if any open cover of X has a finite subcover.

Nonstandard characterization of compactness

- **Theorem** (Robinson): A topological space (X, τ) is compact if and only if for each $x \in {}^*X$, there is $y \in X$ so that $x \in \mu(y)$.



vs.



Fast proof of Tychonoff theorem

Theorem 3.6.2 (Tychonoff) *The product of compact spaces is compact.*

Proof If $X = \Pi_{\alpha \in \mathcal{I}} X_\alpha$ and $g \in {}^*X$, then for each standard $\alpha \in \mathcal{I}$, there is an $x_\alpha \in X_\alpha$ with $g(\alpha) \simeq x_\alpha$. (The x_α's are unique if the spaces X_α are Hausdorff.) The element $f \in X$ with $f(\alpha) = x_\alpha$ for each $\alpha \in \mathcal{I}$ is in X and $g \in \mu(f)$. $\square$

(see Loeb & Wolff)

Continua

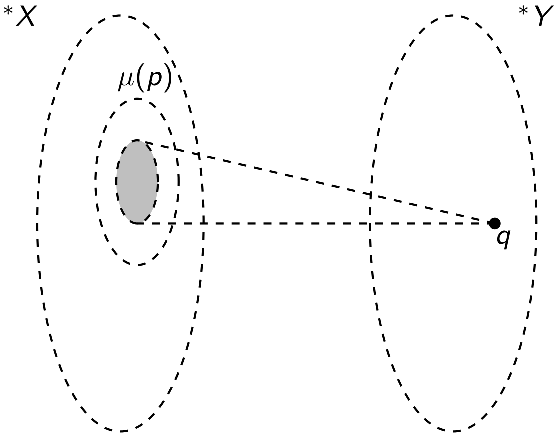
A **continuum** is a compact, connected topological space.
Examples: arc, triod, pseudoarc, etc

Let $\epsilon > 0$. A map $f : X \to Y$ is called an ϵ-map provided that for all $y \in Y$, $\mathrm{diam} f^{-1}[y] < \epsilon$.

We say a continuum X is "Y-like" provided that for all $\epsilon > 0$ there is an ϵ-map from X onto Y.

Small maps

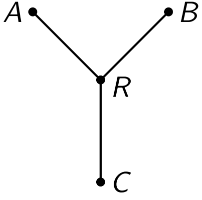
Let $f : {}^*X \to {}^*Y$ be an internal mapping. We say f is **small** provided that for each $q \in {}^*Y$, $f^{-1}[q]$ is contained in the monad of some $p \in {}^*X$.



Transfer: X is Y-like iff there is a small map $f : {}^*X \to {}^*Y$

Arc is triod-like

Let $X = [0, 1]$ and let $p \in \mu(1)$ and pick $q \in \mu(1)$ in the component containing 1. Consider the triod



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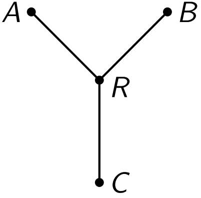
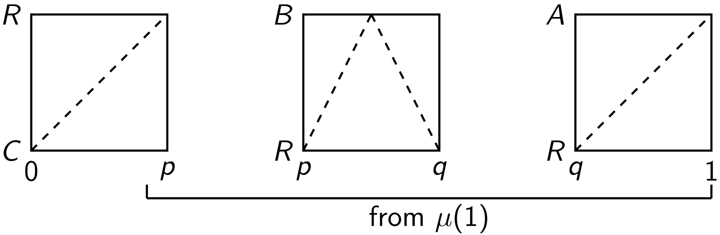
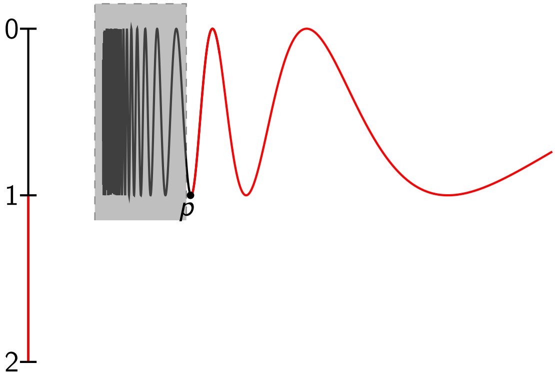


The following map defines a small map $f\colon {}^*[0,1] \to {}^*Y$:



Topologist's sine curve is arclike

Let $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ and consider the point $p = \left(\frac{2}{(2\nu+1)\pi}, -1\right)$, a local minimum in the monad of the limit bar. We define the map from *topologist's sine curve onto $^*[0,2]$ by the following projection:



Thanks

Thanks for attending!

Reference

P. A. Loeb (ed.) and *M. Wolff* (ed.), Nonstandard analysis for the working mathematician. Dordrecht: Kluwer Academic Publishers.