Discrete special functions

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Difference calculus

Forward difference:

$$\Delta f(t) = f(t+1) - f(t)$$

Backward difference:

$$\nabla f(t) = f(t) - f(t-1)$$

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$$\nabla f(t) = f(t) - f(t-1)$$

	Continuous	Discrete
Sum Rule	$(f+g)' = f' + g'$	$\Delta(f+g) = \Delta f + \Delta g$
Product Rule	$(fg)' = f'g + g'f$	$\Delta(fg) = g\Delta f + f^\sigma \Delta g$
Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	$\Delta\left(\frac{f}{g}\right) = \frac{g\Delta f - f\Delta g}{gg^\sigma}$

Discrete polynomials

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$$(a)_k = a(a+1)(a+2)\dots(a+k-1).$$

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(*proof left to audience*)
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Also:

$$t(-1)^n \underbrace{(-(t-1))_n}_{\text{shifted in } t} = (-1)^{n+1}(-t)_{n+1}$$

Classical hypergeometric series

For $p, q \in \{0, 1, 2, \ldots\}$:

$$_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; t\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{t^k}{k!}$$

Classical hypergeometric series

	Function	${}_p\mathcal{F}_q$ representation
Exponential	$e^{\alpha t}$	${}_0\mathcal{F}_0(;;\alpha t)$
Cosine	$\cos(at)$	${}_0\mathcal{F}_1\left(;\frac{1}{2};-\frac{(at)^2}{4}\right)$
Sine	$\sin(at)$	$at{}_0\mathcal{F}_1\left(;\frac{3}{2}:-\frac{(at)^2}{4}\right)$
Bessel	$J_n(t)$	$\frac{z^n}{2^n\Gamma(n+1)}{}_0\mathcal{F}_1\left(;n+1;-\frac{t^2}{4}\right)$
Polylogarithm	$\mathrm{Li}_m(t)$	$t_{m+1}\mathcal{F}_m(1,1,\ldots,1;2,2,\ldots,2;t)$
Sine integral	$\mathrm{Si}(t)$	$t_1\mathcal{F}_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{t^2}{4}\right)$
Assoc. Laguerre poly	$L_n^{(\alpha)}(t)$	$\frac{(\alpha+1)_n}{n!}{}_1\mathcal{F}_1(-n;\alpha+1;t)$
$\vdots$	$\vdots$	$\vdots$

Discrete analogues

Fundamental trick:

$$\text{replace } t^m \text{ with } (-1)^m(-t)_m$$

and

$$\text{replace } tf(t) \text{ with } t\,\underbrace{f(t-1)}_{\text{shifted in } t}$$

Next to equations with constant coefficients, which we studied in § 4, Chap. I, the simplest linear homogeneous difference equations of higher than the first order are those with linear coefficients. The second order equation of this type, namely

$$(99)\qquad (a_2 x + b_2)\, y(x+2) + (a_1 x + b_1)\, y(x+1) + (a_0 x + b_0)\, y(x) = 0,$$

is called the *hypergeometric difference equation*, because, as we shall see in § 7, its solutions can be expressed in terms of the hypergeometric series

$$(100)\quad F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{1\cdot\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\,\gamma(\gamma+1)} z^2 + \cdots.$$

Discrete hypergeometric series – prior notions

Batchelder (1916): "hypergeometric difference equation"

$$(a_2x + b_2)y(x+2) + (a_1x + b_1)y(x+1) + (a_0x + b_0)y(x) = 0.$$

Difference equations of hypergeometric type:

$$\sigma(t)\Delta\nabla y(t) + \tau(t)\Delta y(t) + \lambda y(t) = 0.$$

Khan (1994): "discrete hypergeometric function" (a q-analogue)

$${}_rM_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) = \sum_{k=0}^{\infty} \frac{(q^{a_1})_k \ldots (q^{a_2})_k z^{(k)}}{(q)_k (q^{b_1})_k \ldots (q^{b_s})_k}.$$

Discrete hypergeometric series

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; t, n, \xi) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{\xi^k (-1)^{nk} (-t)_{nk}}{k!} $$

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t,n,\xi) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{\xi^k (-1)^{nk} (-t)_{nk}}{k!} $$

Relationship between ${}_pF_q$ and ${}_p\mathcal{F}_q$:

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t,n,\xi) $$
$$ = {}_{p+n}\mathcal{F}_q\left(a_1,\ldots,a_p,\frac{-t}{n},\frac{-t+1}{n},\ldots,\frac{-t+n-1}{n};b_1,\ldots,b_q;(-n\xi)^n\right) $$

Convergence

Let $y(t) = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; t)$.

Theorem: $y(t)$ **always** converges for all $t \in \mathbb{N}$. If $t \in \mathbb{C} \setminus \mathbb{N}$ and $\xi \in \mathbb{C}$, then $y(t)$ converges whenever $p + n < q + 1$ or both $p + n = q + 1$ and $|\xi| n^n < 1$. If $p + n > q + 1$ or both $p + n = q + 1$ and $|\xi| n^n > 1$, then $y(t)$ diverges. If $p + n = q + 1$ and $|\xi| n^n = 1$, then $y(t)$ converges provided that $\mathrm{Re}\left(\sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k - \frac{1}{n} \sum_{k=0}^{n-1} -t + k \right) > 0$.

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Proof: Routine application of ratio test plus citing a well-known theorem about ${}_p\mathcal{F}_q$.

New special functions

Classic Function	Discrete ${}_pF_q$-analogue
$e^{\alpha t}$	${}_0F_0(;;t,1,\alpha)$
$\cos(at)$	${}_0F_1\left(;\frac{1}{2};t,2,-\frac{a^2}{4}\right)$
$\sin(at)$	$at{}_0F_1\left(;\frac{3}{2}:\underbrace{t-1}_{\text{shift}},2,-\frac{a^2}{4}\right)$
$J_n(t)$	$\frac{(-1)^n(-t)_n}{2^n\Gamma(n+1)}{}_0F_1\left(;n+1;t-n;2,-\frac{1}{4}\right)$
$\mathrm{Li}_m(t,n,1)$	$t_{m+1}F_m(1,1,\ldots,1;2,2,\ldots,2;t-1,n,1)$
$\mathrm{Si}(t)$	$t_1F_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};\underbrace{t-1}_{\text{shift}},2,4\right)$
$L_n^{(\alpha)}(t)$	$\frac{(\alpha+1)_n}{n!}{}_1F_1(-n;\alpha+1;t,1,1)$
$\vdots$	$\vdots$

What properties are preserved? What properties disappear?

Exponential

Classic: ${}_0\mathcal{F}_0(;;\alpha t) = e^{\alpha t}$

Discrete: ${}_0F_0(;;t,1,\alpha) = (1+\alpha)^t$

Classic Property	Discrete Property
Solves $y' = y, y(0) = 1$	Solves $\Delta y(t) = y(t), y(0) = 1$
Positive everywhere	Positive everywhere iff $1+\alpha > 0$
Laplace transform: $\dfrac{1}{z-\alpha}$	$\mathbb{Z}$-Laplace transform $\dfrac{1}{z-\alpha}$

New special functions

<u>Cosine</u>

Classic: $\cos(\alpha t) = {}_0\mathcal{F}_1\left(;\frac{1}{2};-\frac{\alpha^2 t^2}{4}\right)$

Discrete: $\cos_\alpha(t) = {}_0F_1\left(;\frac{1}{2};t,2,-\frac{\alpha^2}{4}\right)$

Classic Property	Discrete Property
Solves $y'' = -y, \begin{cases} y(0)=1 \\ y'(0)=0 \end{cases}$	Solves $\Delta^2 y(t) = y(t), \begin{cases} y(0)=1 \\ \Delta y(0)=0 \end{cases}$
Bounded in $[-1,1]$	Unbounded
Oscillatory	Oscillatory
Laplace transform: $\frac{1}{z-\alpha}$	$\mathbb{Z}$-Laplace transform $\frac{1}{z-\alpha}$

New special functions

<u>Sine</u>

Classic: $\sin(\alpha t) = \alpha t_0\mathcal{F}_1\left(;\frac{3}{2};-\frac{\alpha^2 t^2}{4}\right)$

Discrete: $\sin_\alpha(t) = \alpha t_0 F_1\left(;\frac{3}{2};\underbrace{t-1}_{\text{shift}},2,-\frac{\alpha^2}{4}\right)$

Classic Property	Discrete Property
Solves $y'' = -y, \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$	Solves $\Delta^2 y(t) = y(t), \begin{cases} y(0) = 0 \\ \Delta y(0) = 1 \end{cases}$
Bounded in $[-1,1]$	Unbounded
Oscillatory	Oscillatory
Laplace transform: $\frac{1}{z-\alpha}$	$\mathbb{Z}$-Laplace transform $\frac{1}{z-\alpha}$
$\cos^2(\alpha t) + \sin^2(\alpha t) = 1$	$\cos_\alpha^2(t) + \sin_\alpha^2(t) = e_{\alpha^2}(t)$

Bessel

Classic: $\mathcal{J}_n(t) = \dfrac{t^n}{2^n\Gamma(n+1)} {}_0\mathcal{F}_1\left(;n+1;-\dfrac{t^2}{4}\right)$

Discrete: $J_n(t) = \dfrac{(-1)^n(-t)_n}{2^n\Gamma(n+1)} {}_0\mathcal{F}_1\left(;n+1;\underbrace{t-n}_{\text{shift}};2,-\dfrac{1}{4}\right)$

Classic Property	Discrete Property
$t^2y'' + ty' + (t^2 - n^2)y = 0$	$(-t)_2\Delta^2 y(t-2) + t\Delta y(t-1) + (-t)_2 y(t-2) - n^2 y(t) = 0$
Oscillatory	Oscillatory
Bounded, decays to zero	Unbounded
$\mathscr{L}$ Transform: $\frac{[\sqrt{z^2+1}+z]^{-n}}{\sqrt{z^2+1}}$	$\mathbb{Z}$-Laplace Transform: $\frac{[\sqrt{z^2+1}+z]^{-n}}{\sqrt{z^2+1}}$

Dilogarithm

Classic: $\mathcal{L}i_2(t) = -\int_0^t \frac{\ln(1-\tau)}{\tau}\mathrm{d}\tau = t\,{}_3\mathcal{F}_2(1,1,1;2,2;t) = \sum_{k=1}^{\infty}\frac{t^k}{k^2}$

Discrete: $\mathrm{Li}_2(t,n,\xi) = t\,{}_3F_2(1,1,1;2,2;\underbrace{t-1}_{\text{shift}},n,\xi)$

Classic Property	Discrete Property
$(1-t)ty'' + (1-t)y' = 1$	$t\Delta^2 y(t-1) - (-t)_2\Delta^2 y(t-2) + \Delta y(t) - t\Delta y(t-1) = 1$
$\mathcal{L}i_2(t) + \mathcal{L}i_2(-t) = \frac{1}{2}\mathcal{L}i_2(t^2)$	$\mathrm{Li}_2(t,1,1) + \mathrm{Li}_2(t,1,-1) = \frac{1}{2}\mathrm{Li}_2(t,2,1)$
$\mathrm{Li}_2(1-z) + \mathrm{Li}_2\left(1-\frac{1}{z}\right) = -\frac{\ln^2(z)}{2}$	???
$\mathrm{Li}_2(2) = \frac{\pi^2}{4} - i\pi\ln(2)$	???

Open questions galore

1. Pick your favorite special function.
2. Find its hypergeometric representation (e.g. in a reference like Abramowitz and Stegun).
3. Carry over parameters and coefficients, keeping in mind the "fundamental trick".
4. Explore!

References

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Thanks

Thanks for your attention!
`http://tomcuchta.com` (slides will be posted)