Experiments Related to the Riemann Zeta Function

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What is a function?

$$f(x) = x$$
$$f : \mathbb{R} \rightarrow \mathbb{R}$$

one-dimensional input $\rightarrow$ one-dimensional output

The Imaginary Number

Recall the imaginary number i from Algebra.

$$i = \sqrt{-1}$$

Notice that the powers of i are cyclic.

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

Complex Numbers

Complex numbers have the form

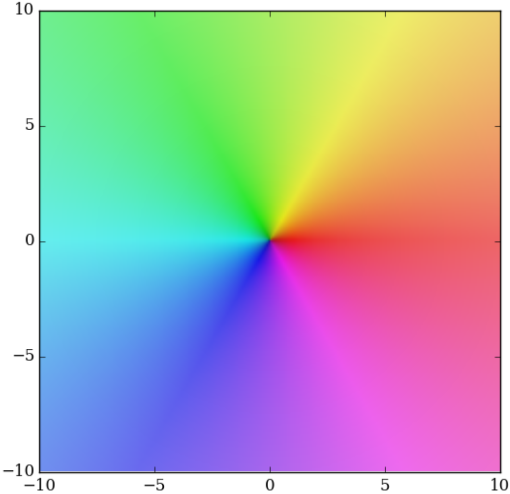
$a + bi$ where $a, b \in \mathbb{R}$

Complex-Valued Functions

$$f(z) = z$$
$$f : \mathbb{C} \to \mathbb{C}$$

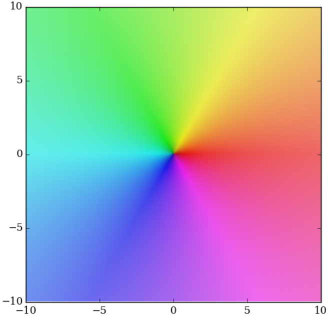
two-dimensional input $\to$ two-dimensional output

Color Scheme

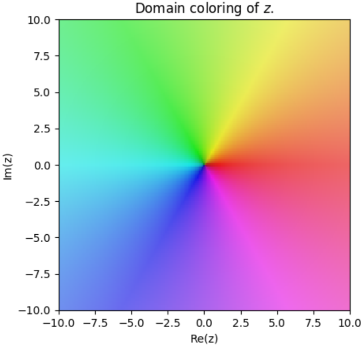


Coloring the identity map

Let $f(z) = z$.



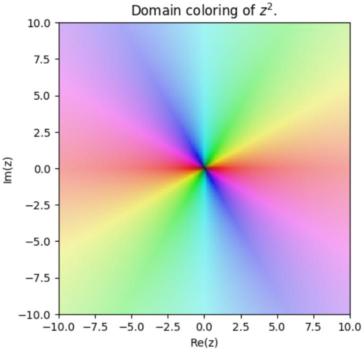
$$\xrightarrow{f}$$



Coloring the squaring map

Let $f(z) = z^2$.

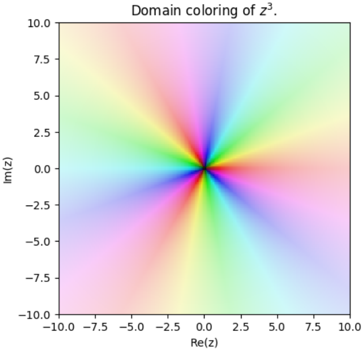
$$\xrightarrow{f}$$



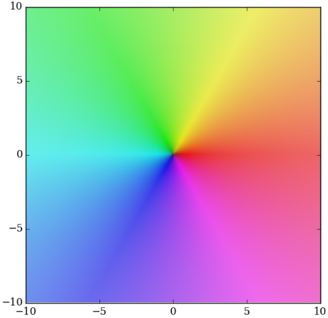
Coloring the cubing map

Let $f(z) = z^3$.

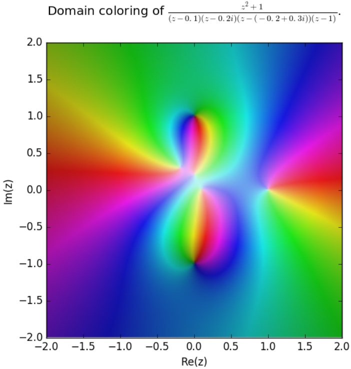
$$\xrightarrow{f}$$



Consider $f(z) = \dfrac{z^2+1}{(z-0.1)(z-0.2i)(z-(-0.2+0.3i))(z-1)}$.



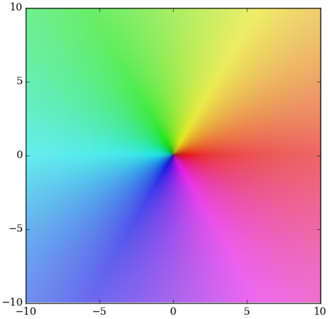
$$\xrightarrow{\ f\ }$$



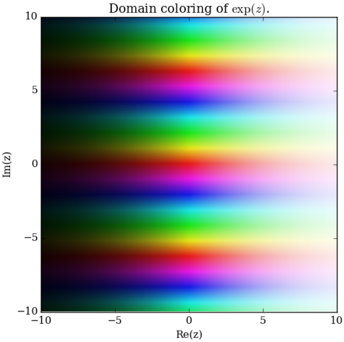
Notice zeros versus poles.

The exponential function e^z is defined by

$$e^z \equiv \exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$



$$\xrightarrow{\exp}$$

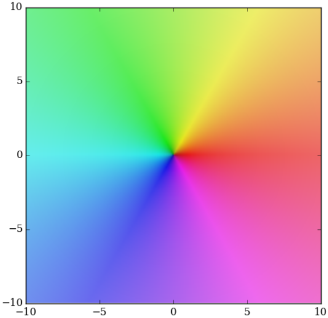


Coloring $\sin(z)$

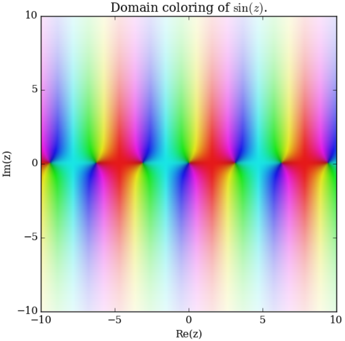
The sine function is defined by

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}.$$

It is extended to the rest of $\mathbb{C}$ by analytic continuation.



$$\xrightarrow{\sin}$$



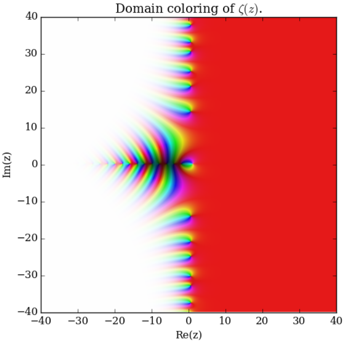
Coloring $\zeta(z)$

The Riemann zeta function is defined (for $\mathrm{Re}(z) > 1$) by

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}.$$

It is extended to the rest of $\mathbb{C}$ by analytic continuation.

$$\xrightarrow{\ \zeta\ }$$



Zeta-like functions

General idea: for some sequence a_k we define

$$\zeta_{a_k}(z) = \sum_{k=1}^{\infty} \frac{1}{{a_k}^z}$$

Notice that if $a_k = k$, then

$$\zeta_{a_k}(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} = \zeta(z),$$

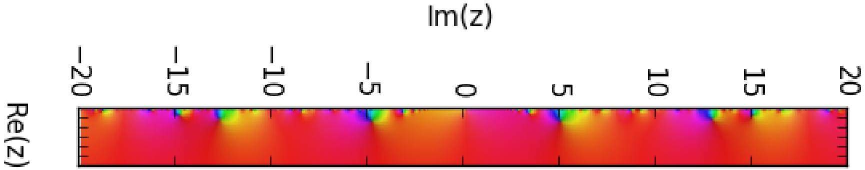
and if $a_k = k^m$, then

$$\zeta_{a_k}(z) = \sum_{k=1}^{\infty} \frac{1}{(k^m)^z} = \sum_{k=1}^{\infty} \frac{1}{k^{mz}} = \zeta(mz),$$

Zeta-like factorial function

If $a_k = k!$ then,

$$\zeta_{a_k}(z) = \sum_{k=1}^{\infty} \frac{1}{(k!)^z},$$



Zeta-like powers of 2 function

Let $a_k = 2^k$. Calculate

$$\begin{aligned}
\zeta_{a_k}(z) &= \sum_{k=1}^{\infty} \frac{1}{(2^k)^z} \\
&= \sum_{k=1}^{\infty} \frac{1}{(2^z)^k} \\
&= \sum_{k=1}^{\infty} (\frac{1}{2^z})^k \\
(g.s.) &= \frac{1/2^z}{1 - 1/2^z} \\
&= \frac{1}{2^z - 1}
\end{aligned}$$

Zeta-like powers of 2 function

Since $\frac{1}{2^z - 1}$ does not have zeros, we will look at its poles instead.

$$\begin{aligned} 2^z - 1 &= 0 \\ 2^{a+bi} &= 1 \\ 2^a 2^{bi} &= 1 \\ 2^{bi} &= 2^{-a} \\ e^{ln(2^{bi})} &= 2^{-a} \\ e^{ibln(2)} &= 2^{-a} \end{aligned}$$

Now solve for b using Euler's formula: $e^{i\theta} = cos(\theta) + isin(\theta)$, where $\theta = bln(2)$.

$$cos(bln(2)) + isin(bln(2)) = 2^{-a}$$

Zeta-like powers of 2 function

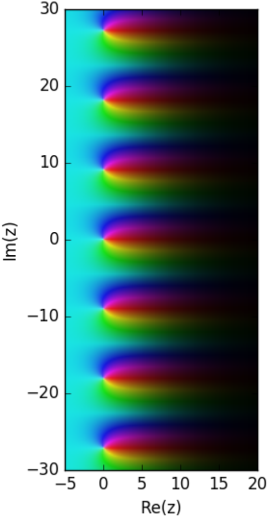
$$cos(bln(2)) + \underbrace{isin(bln(2))}_{=0} = 2^{-a}$$

$$\begin{aligned} \therefore bln(2) &= n\pi \\ b &= \frac{n\pi}{ln(2)} \\ \text{(so cos is positive)} b &= \frac{2n\pi}{ln(2)} \end{aligned}$$

$$\begin{aligned} cos(\frac{2n\pi}{ln(2)} ln(2)) &= 2^{-a} \\ cos(2n\pi) &= 2^{-a} \\ 1 &= 2^{-a} \\ ln(1) &= -aln(2) \\ a &= 0 \end{aligned}$$

Zeta-like powers of 2 function

So, the poles are at $z = a + bi$ where $a = 0$ and $b = \dfrac{2n\pi}{ln(2)}$.



Zeta-like powers of 1+i function

Let $a_k = (1+i)^k$. Using the same method as before, we will end up with

$$\zeta_{a_k}(z) \quad = \frac{1}{(1+i)^z - 1}$$

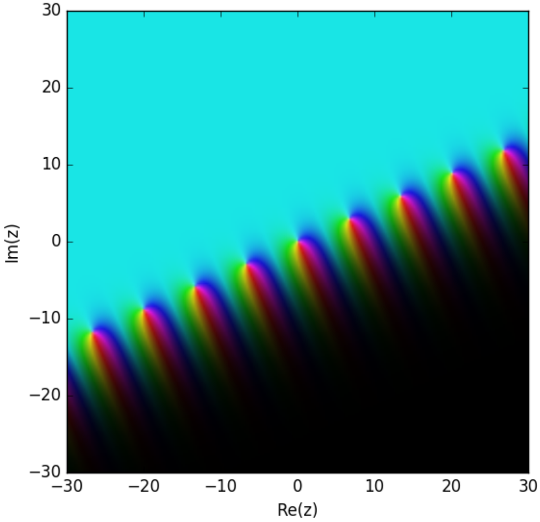
We found that the closed form for the poles of this function is

$$z = \frac{2i\pi n}{ln(1+i)}$$

Using $ln(z) = ln(|z|) + iarg(z)$, we get

$$z = \frac{2i\pi n}{ln(\sqrt{2}) + \frac{i\pi}{4}} \overset{algebra}{=} (\frac{8\pi^2}{4ln(2)^2 + \pi^2} + i\frac{16\pi ln(2)}{4ln(2)^2 + \pi^2})n$$

Zeta-like powers of 1+i function



Thanks

Thank you for attending!