Discrete Sine Integral

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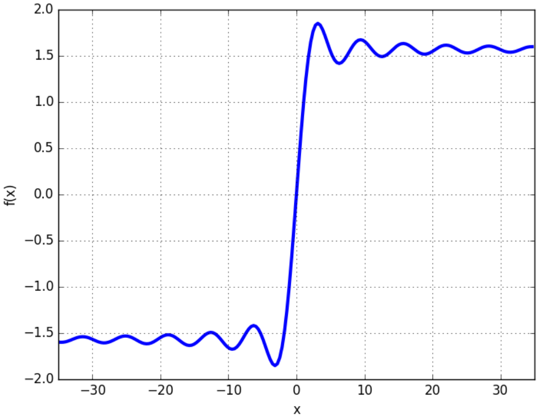
Classic Sine Integral

$$\mathcal{SI}(x) = \int_0^x \frac{\sin t}{t}\,dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)(2k-1)!} \tag{1}$$



Differential Equation of Classic Sine Integral

$\mathcal{SI}$ Satisfies the following differential equation:

$$ty'''(t) + 2y''(t) + ty'(t) = 0 \tag{2}$$

Note: this function has applications within signal processing (e.g. step response of the sinc filter) and electronics.

Discrete Calculus – Discrete Derivative

The forward difference operator Δ is defined by:

$$\Delta f(x) = f(x+1) - f(x) \tag{3}$$

The rising factorial $(a)_k$ is defined by:

$$(a)_k = a(a+1)(a+2)\ldots(a+k-1) \tag{4}$$

The discrete monomials are defined by:

$$(-1)^k(-t)_k \tag{5}$$

Discrete Calculus – Discrete Integral

We define the discrete integral "from a to b" of a function f by

$$\sum_{k=a}^{b-1} f(k) \tag{6}$$

because a fundamental theorem of discrete calculus holds:

$$\Delta \left(\sum_{k=a}^{x-1} f(k) \right) = f(x) \tag{7}$$

$$\sum_{k=a}^{b-1} \Delta f(k) = f(b) - f(a) \tag{8}$$

Discrete Calculus – Discrete Power Rule

Theorem: (Discrete Power Rule – Discrete Difference)

$$\Delta[(-1)^k(-t)_k] = k[(-1)^{k-1}(-t)_{k-1}] \tag{9}$$

Proof:

$$\begin{aligned}
&\Delta[(-1)^k(-t)_k] = (-1)^k(-(t+1))_k - (-1)^k(-t)_k \\
&= [(-1)^{k-1}(-t)_{k-1}][(-1)(-t-1) - (-1)(-t+k-1)] \\
&= [(-1)^{k-1}(-t)_{k-1}][t+1-t-1+k] \\
&= k[(-1)^{k-1}(-t)_{k-1}]
\end{aligned}$$

Theorem: (Discrete Power Rule – Discrete Integral)

$$\sum_{k=0}^{t-1}(-1)^n(-k)_n = \frac{(-1)^{n+1}(-t)_{n+1}}{n+1} \tag{10}$$

Discrete Calculus – Discrete Sine

The discrete exponential $e_p(t)$ is defined by difference equation

$$\Delta y(t) = p(t)y(t), \quad y(0) = 1 \tag{11}$$

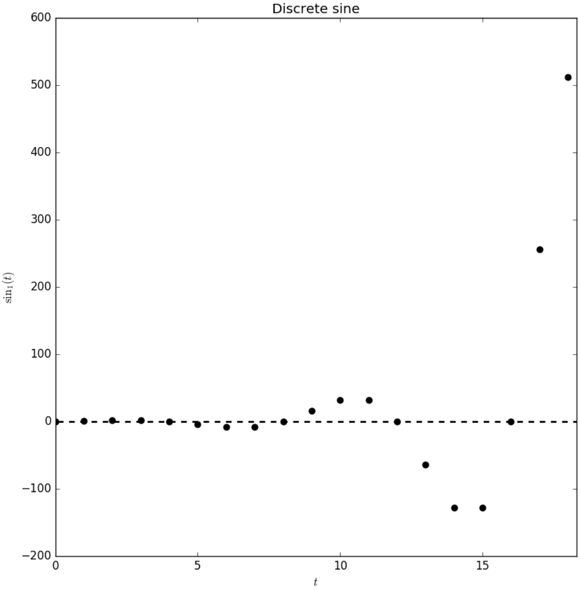
The discrete sine is defined by

$$\sin_1(t) = \frac{e_i(t) - e_{-i}(t)}{2i} \tag{12}$$

and obeys the difference equation

$$\Delta^2 y(t) = -y(t), \quad y(0) = 0 \tag{13}$$

Discrete Calculus – Discrete Sine



Discrete Calculus – Shifting

It can be shown that if

$$f(t) = (-1)^n (-t)_n \tag{14}$$

then

$$tf(t-1) = (-1)^{n+1} (-t)_{n+1} \tag{15}$$

and

$$\frac{f(t+1)}{t+1} = (-1)^{n-1} (-t)_{n-1} \tag{16}$$

Discrete Sine Integral Difference Equation

The classical sine integral obeys the following differential equation:

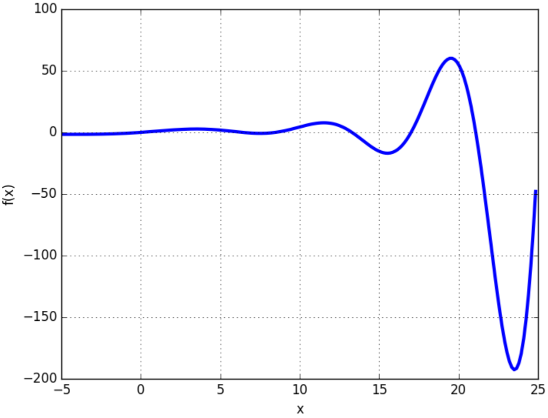
$$ty'''(t) + 2y''(t) + ty'(t) = 0.$$

We propose the following analagous difference equation to define a discrete sine integral function:

$$\mathrm{t}\Delta^3\mathrm{Si}(\mathrm{t}-1) + 2\Delta^2\mathrm{Si}(\mathrm{t}) + \mathrm{t}\Delta\mathrm{Si}(\mathrm{t}-1) = 0. \quad (17)$$

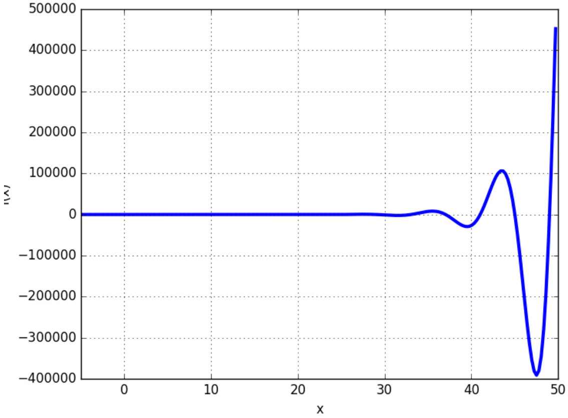
Discrete Sine Integral

$$\mathrm{Si}(t) = \sum_{k=0}^{x-1} \frac{\sin_1(k+1)}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}[(-1)^{2k-1}(-t)_{2k-1}]}{(2k-1)(2k-1)!} \qquad (18)$$



Discrete Sine Integral

$$\mathrm{Si}(t) = \sum_{k=0}^{x-1} \frac{\sin_1(k+1)}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}[(-1)^{2k-1}(-t)_{2k-1}]}{(2k-1)(2k-1)!}$$



Solution of Difference Equation

$$\mathrm{t}\Delta^3\mathrm{Si}(\mathrm{t}-1)+2\Delta^2\mathrm{Si}(\mathrm{t})+\mathrm{t}\Delta\mathrm{Si}(\mathrm{t}-1)=0$$

$$\begin{aligned}
\Delta\mathrm{Si}(t) &= \Delta\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}[(-1)^{2k-1}(-t)_{2k-1}]}{(2k-1)(2k-1)!}\right] \\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)(2k-1)!}\Delta\left[(-1)^{2k-1}(-t)_{2k-1}\right] \\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)(2k-1)!}(2k-1)\left[(-1)^{2k-2}(-t)_{2k-2}\right] \\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(-1)^{2k-2}(-t)_{2k-2}}{(2k-1)!}
\end{aligned}$$

Solution of Difference Equation

$$\mathrm{t}\Delta^3\mathrm{Si}(\mathrm{t}-1) + 2\Delta^2\mathrm{Si}(\mathrm{t}) + \mathrm{t}\Delta\mathrm{Si}(\mathrm{t}-1) = 0$$

$$\begin{aligned}
t\Delta\mathrm{Si}(t-1) \quad &= t\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\left[(-1)^{2k-2}(-t+1)_{2k-2}\right] \\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\left[(-1)^{2k-1}(-t)_{2k-1}\right] \\
&= \sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{(2k-3)!}\left[(-1)^{2k-3}(-t)_{2k-3}\right]
\end{aligned}$$

Solution of Difference Equation

$$\mathrm{t}\Delta^3\mathrm{Si}(\mathrm{t}-1) + 2\Delta^2\mathrm{Si}(\mathrm{t}) + \mathrm{t}\Delta\mathrm{Si}(\mathrm{t}-1) = 0.$$

$$\Delta^2\mathrm{Si}(t) = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}(2k-2)[(-1)^{2k-3}(-t)_{2k-3}]}{(2k-1)!}$$

Solution of Difference Equation

$$\mathrm{t}\Delta^3\mathrm{Si}(\mathrm{t}-1) + 2\Delta^2\mathrm{Si}(\mathrm{t}) + \mathrm{t}\Delta\mathrm{Si}(\mathrm{t}-1) = 0.$$

$$\Delta^3\mathrm{Si}(t) \quad = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}(2k-2)(2k-3)[(-1)^{2k-4}(-t)_{2k-4}]}{(2k-1)!}$$

$$\begin{aligned}
&t\Delta^3\mathrm{Si}(t-1) \\
&\quad = t\sum_{k=2}^{\infty} \frac{(-1)^{k-1}(2k-2)(2k-3)}{(2k-1)!}\left[(-1)^{2k-4}(-t+1)_{2k-4}\right] \\
&\quad = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}(2k-2)(2k-3)}{(2k-1)!}\left[(-1)^{2k-3}(-t)_{2k-3}\right]
\end{aligned}$$

Solution of Difference Equation

$$\mathrm{t}\Delta^3\mathrm{Si}(\mathrm{t}-1)+2\Delta^2\mathrm{Si}(\mathrm{t})+\mathrm{t}\Delta\mathrm{Si}(\mathrm{t}-1)=0.$$

Combining these results:

$$\sum_{k=2}^{\infty}\left[\frac{(-1)^{k-1}(2k-2)(2k-3)}{(2k-1)!}\right.$$

$$\left.+\frac{2(-1)^{k-1}(2k-2)}{(2k-1)!}+\frac{(-1)^{k-2}}{(2k-3)!}\right]\left[(-1)^{2k-3}(-t)_{2k-3}\right]$$

Solution of Difference Equation

$$
\begin{aligned}
&\left[\frac{(-1)^{k-1}(2k-2)(2k-3)}{(2k-1)!} + \frac{2(-1)^{k-1}(2k-2)}{(2k-1)!} + \frac{(-1)^{k-2}}{(2k-3)!}\right] \\
&= \frac{(-1)^{k-1}}{(2k-1)!}\left[(2k-2)(2k-3) + 2(2k-2) - (2k-2)(2k-1)\right] \\
&= \frac{(-1)^{k-1}}{(2k-1)!}\left[(4k^2-6k-4k+6) + (4k-4) - (4k^2-2k-4k+2)\right] \\
&= 0,
\end{aligned}
$$

proving that Si satisfies the difference equation

$$
\mathrm{t}\Delta^3\mathrm{Si}(\mathrm{t}-1) + 2\Delta^2\mathrm{Si}(\mathrm{t}) + \mathrm{t}\Delta\mathrm{Si}(\mathrm{t}-1) = 0.
$$

Future Work

More properties of discrete sine integral
Extend theory to discrete exponential integrals
Extend theory to discrete cosine integrals

Sayonara

Thank you for coming